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A FULLY NONLINEAR THEORY OF CURVED AND TWISTED COMPOSITE ROTOR BLADES ACCOUNTING FOR WARPINGS AND THREE-DIMENSIONAL STRESS EFFECTS

PERNGJIN F. PAI and ALI H. NAYFEH
Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, U.S.A.

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Abstract—A new approach is used to develop a geometrically exact nonlinear beam model for naturally curved and twisted solid composite rotor blades undergoing large vibrations in three-dimensional space. A combination of the new concepts of local displacements and local engineering stresses and strains, a new interpretation and manipulation of the virtual local rotations, an exact coordinate transformation, and the extended Hamilton principle is used to derive six fully nonlinear equations of motion describing one extension, two bending, one torsion, and two shearing vibrations of composite beams. The formulation is based on an energy approach, but the derivation is fully correlated with the Newtonian approach and provides a straightforward explanation of all nonlinear structural terms without using complex tensor operations or asymptotic expansions. The theory accounts for in-plane and out-of-plane warpings due to bending, extensional, shearing and torsional loadings, elastic couplings among warpings, and three-dimensional stress effects by using the results of a two-dimensional, static, sectional, finite element analysis. Also, the theory fully accounts for extensionality, initial curvatures and geometric nonlinearities. The equations display linear elastic couplings due to structural anisotropy and initial curvatures and nonlinear geometric couplings. The theory contains most of the existing beam theories as special cases, and the final equations of motion are put in compact matrix form.

NOMENCLATURE

abc	a reference rectangular frame fixed on the hub
A_u, A_v, A_w	translational inertia terms
$A_{\theta_1}, A_{\theta_2}, A_{\theta_3}, A_{\rho_1}, A_{\gamma_5}, A_{\gamma_6}$	rotational inertia terms
$\bar{a}\bar{b}\bar{c}$	abc when $t = 0$
$\mathbf{D}$	the absolute displacement vector of the observed point
e	the axial strain on the reference line
$\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$	internal stress resultants
I_{ij}, $[I_1]$, $[I_2]$, $[I_3]$	mass inertias
$\mathbf{i}_a, \mathbf{i}_b, \mathbf{i}_c$	unit vectors along a, b, c, $\{\mathbf{i}_{abc}\} = \{\mathbf{i}_a, \mathbf{i}_b, \mathbf{i}_c\}^T$
$\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z$	unit vectors along x, y, z, $\{\mathbf{i}_{xyz}\} = \{\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z\}^T$
$\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$	unit vectors along ξ, η, ζ, $\{\mathbf{i}_{123}\} = \{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}^T$
k_1, k_2, k_3	the initial curvatures
$[k]$	the undeformed curvature matrix, $[k] = [P(k_1, k_2, k_3)]$
$\tilde{M}_1, \tilde{M}_2, \tilde{M}_3$	internal moments
$[\tilde{Q}]$	the material stiffness matrix
$\mathbf{R}$	$= A(s)\mathbf{i}_a + B(s)\mathbf{i}_b + C(s)\mathbf{i}_c$, the undeformed position vector of the reference point
s	the undeformed arc length from the beam root to the reference point, $(\cdot)' = \partial(\cdot)/\partial s$
t	time, $(\dot{\,}) = \partial(\cdot)/\partial t$
$[T]$	transformation matrix, $\{\mathbf{i}_{123}\} = [T]\{\mathbf{i}_{xyz}\}$
$[T^x]$	transformation matrix, $\{\mathbf{i}_{xyz}\} = [T^x]\{\mathbf{i}_{abc}\}$
$[T^a]$	transformation matrix, $\{\mathbf{i}_{abc}\} = [T^a]\{\mathbf{i}_{\bar{a}\bar{b}\bar{c}}\}$
$\mathbf{U}$	$= u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3$, the local displacement vector of the observed point
$U(t)$, $V(t)$, $W(t)$	the displacement components of the rotor hub
u, v, w	the displacements of the reference point with respect to the axes x, y, z
W_1, W_2, W_3	warpings with respect to the axes ξ, η, ζ, $\{W\} = \{W_1, W_2, W_3\}^T$
xyz	an undeformed orthogonal curvilinear frame
$\overline{xyz}$	xyz when $t = 0$
α	an Euler angle related to the bending of the beam
γ_5, γ_6	shear rotation angles at the reference line
$\delta\theta_1, \delta\theta_2, \delta\theta_3$	local virtual rotation angles of the observed cross section
$\{\varepsilon\}$	$= \{\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{22}, \varepsilon_{23}, \varepsilon_{33}\}^T$, strain vector, $\{\varepsilon\} = [S]\{\psi\} + \{\bar{\psi}\}$
$[\mathrm{K}]$	a deformed curvature matrix, $[\mathrm{K}] = [P(\rho_1, \rho_2, \rho_3)]$
$\xi\eta\zeta$	a deformed local orthogonal curvilinear frame

ρ_1, ρ_2, ρ_3 deformed curvatures
$\{\sigma\}$ $= \{\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33}\}^T$, stress vector, $\{\sigma\} = [\tilde{Q}]\{\varepsilon\}$
ϕ an Euler angle related to the torsion of the beam
$\{\psi\}$ $= \{e, \gamma_6, \gamma_5, \rho_1 - k_1, \rho_2 - k_2, \rho_3 - k_3\}^T$
$\{\bar{\psi}\}$ $= \{y\gamma_6' + z\gamma_5', 0, 0, 0, 0, 0\}^T$
$\{\tilde{\psi}\}$ $= \{e, \gamma_6, \gamma_5, \rho_1 - k_1, \rho_2 - k_2, \rho_3 - k_3, \gamma_6', \gamma_5'\}^T$
$\Omega \mathbf{i}_h$ the angular velocity vector of the rotor hub
$\vec{\omega}$ the angular velocity vector of the coordinate system $\xi\eta\zeta$
$\vec{\omega}^a$ the angular velocity vector of the coordinate system abc
$\vec{\omega}^x$ the angular velocity vector of the coordinate system xyz
$[P(x_1, x_2, x_3)]$ $\equiv \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}$.

1. INTRODUCTION

A nonlinear curved and twisted beam is often used to model helicopter rotor blades, aviation propeller blades, turbine blades, arm-type positioning mechanisms of magnetic disk drives, and robot manipulators; beam-type specimens are usually used for the characterization of materials in experiments. Moreover, the post-buckling strength of beams plays an important role in the design of aircraft structures because conventional aircraft structural elements are often designed to operate in the post-buckling range where nonlinear beam theories are needed. Recently, the rapid development in aerospace exploration has stimulated research into the dynamics and control of flexible composite structures because most space structures are large and flexible and characterized by low inherent dampings and long duration responses to transient loads. Consequently, the development of a general curved beam theory, especially a nonlinear composite beam theory, is a constant research interest.

Geometric nonlinearities are important when the lateral deflections of a beam are finite. In this case, the axial force can play a significant role in carrying the load. Moreover geometric nonlinearities couple the equations governing the extension, bending, torsion and shearing vibrations. They are important in post-buckling analyses, nonlinear panel flutter and dynamic stability problems. Because the ratios of the Young's moduli to the shear moduli are between 20 and 50 in modern composites and between 2.5 and 3.0 in isotropic materials, transverse shear deformations are significant for composite structures, especially for laminated or thick structures (Whitney, 1987; Rao *et al.*, 1976). Moreover, because of the anisotropy, the asymmetry of the cross-section, and the non-uniformity of Poisson's ratios over the cross-section, torsional warping, in-plane warpings and transverse normal stresses can be significant for laminated composite beams or box beams. In other words, all three-dimensional stress effects can be important for general anisotropic beams. Hence, the objective of this paper is to develop a general nonlinear curved beam theory that includes three-dimensional stress effects (i.e. out-of-plane and in-plane warpings, transverse normal stresses, etc.) and geometric nonlinearities as well as anisotropy and initial curvatures.

Von Kármán strains do not fully account for geometric nonlinearities due to large rotations (Pai and Nayfeh, 1992). To fully account for geometric nonlinearities by using the Green–Lagrange strains, the correct conjugate stresses, the second Piola–Kirchhoff stresses must be used, not the Cauchy stresses or engineering stresses. However, even if the second Piola–Kirchhoff stresses are used correctly with Green–Lagrange strains in a formulation, they cannot easily match with real boundary conditions because they are measures of energy, not of geometry, and have no definite directions. Moreover, the material stiffness constants obtained from small-strain experiments cannot be directly used to relate the Green–Lagrange strains and second Piola–Kirchhoff stresses in the constitutive equation; if the material stiffnesses from small-strain experiments are used, geometric nonlinearities can be mistaken for material nonlinearities. Consequently, local engineering stress and strain measures are more appropriate and convenient ones to use.

Recently, two or three successive Euler-like rotations are commonly used in obtaining the exact transformation matrix that relates the deformed and undeformed states and hence accounts for large geometric nonlinearities (Hodges and Dowell, 1974; Hodges, 1976; Dowell *et al.*, 1977; Crespo da Silva and Glynn, 1978; Alkire, 1984; Rosen and Rand, 1986; Rosen *et al.*, 1987; Bauchau and Hong, 1987; Minguet and Dugundji, 1990; Pai and Nayfeh, 1990; Simo and Vu-Quoc, 1991; Banan *et al.*, 1991). However, because finite rotations are not vector quantities, variations of three successive Euler-like angles are not independent because they are not along three perpendicular directions. Moreover, coordinate transformations using three Euler angles result in asymmetric equations of motion, and the torsion-related angle ϕ does not represent the real twist angle even if only two Euler angles are used in the transformation (Pai and Nayfeh, 1990). Hence, the concept of virtual local rotation (Pai and Nayfeh, 1992) is needed in order to derive fully nonlinear equations governing the motions along three perpendicular directions.

Moreover, shear and torsional warpings are structure-dependent functions and are independent of flexural displacements of the reference line of the beam because warpings are displacements with respect to the deformed cross-sections. Hence, warpings need to be modeled by using further dependent variables, which results in more equations of motion. Also, to account for nontrivial shear stress σ_{23} and transverse normal stresses σ_{22} and σ_{33}, where the axis 1 is along the reference line of the beam, some energy-related stress resultants need to be introduced. Furthermore, the elastic energy of anisotropic beams is a very complicated function, especially for beams with significant torsional and shear warpings, and hence it is inconvenient to obtain the explicit expression of elastic energy first and then use an energy formulation to derive the equations of motion.

Since in-plane and out-of-plane warpings are relative, local displacements with respect to the deformed cross-section and are much smaller than global displacements, their influence on inertial forces is negligible. But these displacements offer extra degrees of freedom for the deformation of cross-sections and hence greatfully affect the elastic properties. Consequently, to account for three-dimensional stress effects in a one-dimensional beam model, one needs to include warping effects in the elastic energy and the constitutive equations. For geometrically nonlinear elastic anisotropic beams, a one-dimensional beam model with structural stiffness matrices and warping functions determined by using a two-dimensional sectional analysis is a general and practical approach in solving nonlinear anisotropic beam problems (Hodges, 1988). This approach was used by Bori and Merlini (1986) and Hodges (1990) among others. Since helicopter rotor blades are typically built-up structures with anisotropic and/or nonhomogeneous materials, an analytical approach to obtain the structural stiffness matrices and warping functions is almost impossible and a finite element-based approach is more practical. Giavotto *et al.* (1983) presented a two-dimensional, static, sectional, finite element analysis of warpings for straight beams, the formulation of which is linear and based on an undeformed coordinate system. After Giavotto *et al.* (1983), Altigan and Hodges (1991) and Hodges *et al.* (1992) presented another derivation of such sectional analysis of straight beams.

For initially curved and twisted composite beams, Pai and Nayfeh (1993) used an approach completely different to those used by Giavotto *et al.* (1983), Borri and Merlini (1986), Atilgan and Hodges (1991), and Hodges (1990), to formulate a nonlinear two-dimensional sectional finite element analysis. Moreover, in the two-dimensional sectional analysis one needs to restrain the six rigid-body motions (three translations and three rotations) in order to make the problem nonsingular. Because of the use of the new concept of local stress and strain measures and a different interpretation and manipulation of the virtual local rotations, the six constraint equations derived by Pai and Nayfeh (1993) are exact. On the other hand, the constraints used in the literature are approximate, indirect and inconvenient to implement in finite element computations. The six constraints used by Borri and Merlini (1986) are based on the assumption that the average work done by the surface tractions in producing the warping displacements on the cross-section is zero. Four of the six constraints used by Hodges *et al.* (1992) are based on the assumption that the average warping displacements and in-plane rotation due to warping on the cross-section are zero and the other two constraints are used to define the deformed, local coordinate system.

In this paper, we develop a geometrically exact nonlinear theory for initially curved and twisted composite beams by using a combination of the new concept of local engineering stress and strain measures, new manipulation and interpretation of the concept of virtual local rotations, an exact coordinate transformation, and the extended Hamilton principle. The theory is developed in a very different manner from any other approaches in the literature and provides a straightforward explanation and very clear insight into the physical meanings of all the structural and inertial terms. We extend our former beam theory (Pai and Nayfeh, 1992) by considering beams with arbitrary cross-sections and initial bending and twisting curvatures and accounting for three-dimensional stress effects by using the warping functions obtained from a two-dimensional, sectional, finite element analysis (Pai and Nayfeh, 1993). Extensionality is considered, shear correction factors are not required, couplings due to anisotropy and initial curvatures are included, and the influence of inertial forces due to the rigid-body motion of the beam reference frame is also included. Moreover, the formulations using an energy approach and a Newtonian approach are fully correlated in the derivation.

2. COORDINATE SYSTEMS AND CURVATURES

We consider the naturally curved and twisted beam shown in Fig. 1. We assume that the cross-section can be of any shape and the beam can be nonprismatic. Three coordinate systems are used in the derivation. The xyz system is an orthogonal curvilinear coordinate system, where the x axis denotes the reference line of the beam and s is the undeformed arc length from the root of the beam to the reference point on the observed cross-section. The abc system is a reference rectangular coordinate system fixed on the hub of the beam, where the c axis is along the centerline of the rotor hub. As the formulation will start from a particle point of view, there is no need to calculate the locations of the mass centroid, area centroid, tensile axis, or shear center of the cross-section. Thus, the origin (i.e. the reference point of the observed cross-section) of the xyz system can be at the mass centroid, the area centroid, the shear center, or any other point on the cross-section and the y and z axes need not be the principal axes of the cross-section. Moreover, the $\xi\eta\zeta$ system is a local orthogonal curvilinear coordinate system, where the ξ axis represents the deformed reference line and the η and ζ axes represent the deformed y and z axes only if there were no shear and torsional warpings.

We let $\mathbf{i}_a$, $\mathbf{i}_b$ and $\mathbf{i}_c$ be unit vectors along the a, b, and c axes; $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$ be unit vectors along the x, y and z axes; and $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ be unit vectors along the ξ, η and ζ axes. Moreover, we assume that the $\bar{a}\bar{b}\bar{c}$ and $\overline{xyz}$ systems represent the abc and xyz systems at $t = 0$, t is the time, $\Omega(t)$ is the angular velocity of the hub, and $\mathbf{i}_h$ is the unit vector along the direction of $\Omega(t)$, as shown in Fig. 2.

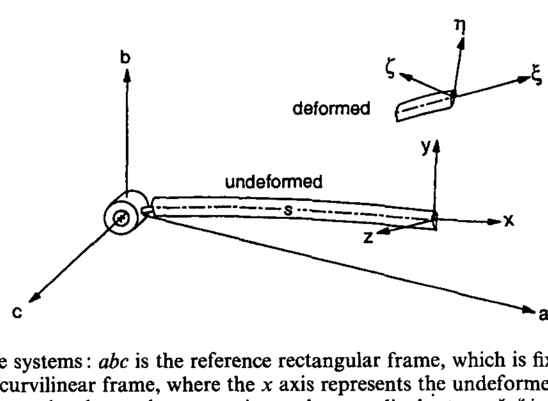

Fig. 1. Coordinate systems: abc is the reference rectangular frame, which is fixed on the hub; xyz is the orthogonal curvilinear frame, where the x axis represents the undeformed reference line and the y and z axes are on the observed cross-section and perpendicular to x; $\xi\eta\zeta$ is the local orthogonal curvilinear coordinate system, where the ξ axis represents the deformed reference line.

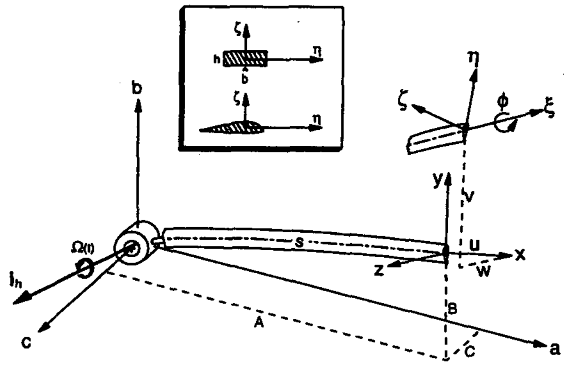

Fig. 2. The system configuration and displacements: u, v and w are displacement components of the reference point on the observed cross-section and ϕ is an Euler angle related to the torsion of the beam.

We let $\mathbf{I}_j$ denote the unit vectors along the axes of an arbitrary orthogonal coordinate system and let its angular velocity be given by

$$\vec{\omega}^I = \omega_1^I \mathbf{I}_1 + \omega_2^I \mathbf{I}_2 + \omega_3^I \mathbf{I}_3. \tag{1a}$$

Then, if $\mathbf{X}$ and $\mathbf{Y}$ are two arbitrary vectors given by

$$\mathbf{X} = X_1\mathbf{I}_1 + X_2\mathbf{I}_2 + X_3\mathbf{I}_3, \quad \mathbf{Y} = Y_1\mathbf{I}_1 + Y_2\mathbf{I}_2 + Y_3\mathbf{I}_3 \tag{1b}$$

we obtain the following identities

$$\mathbf{X}\times\mathbf{Y} = \{\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3\}[P(X)]^T \begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{Bmatrix} = \{\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3\}[P(Y)] \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \tag{1c}$$

$$X_1\dot{\mathbf{I}}_1 + X_2\dot{\mathbf{I}}_2 + X_3\dot{\mathbf{I}}_3 = \vec{\omega}^I \times \mathbf{X} = \{\omega^I\}^T [P(X)]^T \{\mathbf{I}_{123}\} \tag{1d}$$

$$\begin{aligned} X_1\ddot{\mathbf{I}}_1 + X_2\ddot{\mathbf{I}}_2 + X_3\ddot{\mathbf{I}}_3 &= \dot{\vec{\omega}}^I \times \mathbf{X} + \vec{\omega}^I \times (\vec{\omega}^I \times \mathbf{X}) \\ &= \{\dot{\omega}^I\}^T [P(X)]^T \{\mathbf{I}_{123}\} + \{\omega^I\}^T [P(X)]^T [P(\omega^I)] \{\mathbf{I}_{123}\}, \end{aligned} \tag{1e}$$

where

$$[P(X)] \equiv [P(X_1, X_2, X_3)] \equiv \begin{bmatrix} 0 & X_3 & -X_2 \\ -X_3 & 0 & X_1 \\ X_2 & -X_1 & 0 \end{bmatrix}. \tag{1f}$$

In Fig. 2, u, v and w represent the displacement components of the reference point on the observed cross-section with respect to the x, y and z axes. The undeformed position vector $\mathbf{R}$ of the reference point of the observed cross-section is known and given by

$$\mathbf{R} = A(s)\mathbf{i}_a + B(s)\mathbf{i}_b + C(s)\mathbf{i}_c. \tag{2}$$

Also, the undeformed angles θ_{21}, θ_{22} and θ_{23} of the y axis with respect to the abc system are assumed to be known and given by

$$\theta_{21} = \cos^{-1}(\mathbf{i}_y \cdot \mathbf{i}_a), \quad \theta_{22} = \cos^{-1}(\mathbf{i}_y \cdot \mathbf{i}_b), \quad \theta_{23} = \cos^{-1}(\mathbf{i}_y \cdot \mathbf{i}_c), \tag{3}$$

where the θ_{2i} $(i = 1,2,3)$ are functions of s only and $0 \leqslant \theta_{2i} \leqslant 180°$. It follows from eqn (2) that

$$\mathbf{i}_x = \mathbf{R}' = A'\mathbf{i}_a + B'\mathbf{i}_b + C'\mathbf{i}_c, \tag{4}$$

where the prime indicates the derivative with respect to s. Using eqns (3) and (4) and the orthonormality property of the unit vectors, we obtain

$$\{\mathbf{i}_{xyz}\} = [T^x]\{\mathbf{i}_{abc}\}, \tag{5a}$$

where the transformation matrix $[T^x]$ is given by

$$[T^x] = \begin{bmatrix} A' & B' & C' \\ \cos\theta_{21} & \cos\theta_{22} & \cos\theta_{23} \\ B'\cos\theta_{23} - C'\cos\theta_{22} & C'\cos\theta_{21} - A'\cos\theta_{23} & A'\cos\theta_{22} - B'\cos\theta_{21} \end{bmatrix}. \tag{5b}$$

Using eqn (5a), the orthonormality property of $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$ [e.g. $\partial(\mathbf{i}_x \cdot \mathbf{i}_x)/\partial s = \partial(\mathbf{i}_x \cdot \mathbf{i}_y)/\partial s = 0$], and the identity $[T^x]^{-1} = [T^x]^T$ (because $[T^x]$ is a unitary matrix), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\{\mathbf{i}_{xyz}\} = [k]\{\mathbf{i}_{xyz}\}, \tag{6a}$$

where the initial curvature matrix $[k]$ is given by

$$[k] \equiv \frac{\mathrm{d}[T^x]}{\mathrm{d}s}[T^x]^T = [P(k_1, k_2, k_3)] \tag{6b}$$

$$k_1 = \sum_{i=1}^{3} \frac{\mathrm{d}T^x_{2i}}{\mathrm{d}s} T^x_{3i}, \quad k_2 = -\sum_{i=1}^{3} \frac{\mathrm{d}T^x_{1i}}{\mathrm{d}s} T^x_{3i}, \quad k_3 = \sum_{i=1}^{3} \frac{\mathrm{d}T^x_{1i}}{\mathrm{d}s} T^x_{2i}. \tag{6c}$$

Here, k_1, k_2 and k_3 are the initial curvatures with respect to the x, y and z axes, respectively; they are functions of s only.

Following Alkire (1984), we use two sequential Euler angles α and ϕ to describe the rotation of the observed element from the undeformed to the deformed position. The first angle, α, characterizes the bending rotation about the axis n, which is defined later, as shown in Fig. 3. The second angle, ϕ, is related to the torsional motion about the bent reference axis ξ. Hence, the transformation which relates the undeformed coordinate system xyz to the deformed coordinate system $\xi\eta\zeta$ is

$$\{\mathbf{i}_{123}\} = [T]\{\mathbf{i}_{xyz}\}, \tag{7a}$$

where the transformation matrix $[T]$ is given by

$$[T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} [B(\alpha)]. \tag{7b}$$

The transformation matrix $[B(\alpha)]$ is due to the bending rotation α (see Fig. 3), which rotates the x axis to the ξ axis, the y axis to the y_1 axis, and the z axis to the z_1 axis. We note that the angles between the y and y_1 axes and the z and z_1 axes are not equal to α because the y–y_1 and z–z_1 planes are not perpendicular to the n axis. It follows from Fig. 3 that

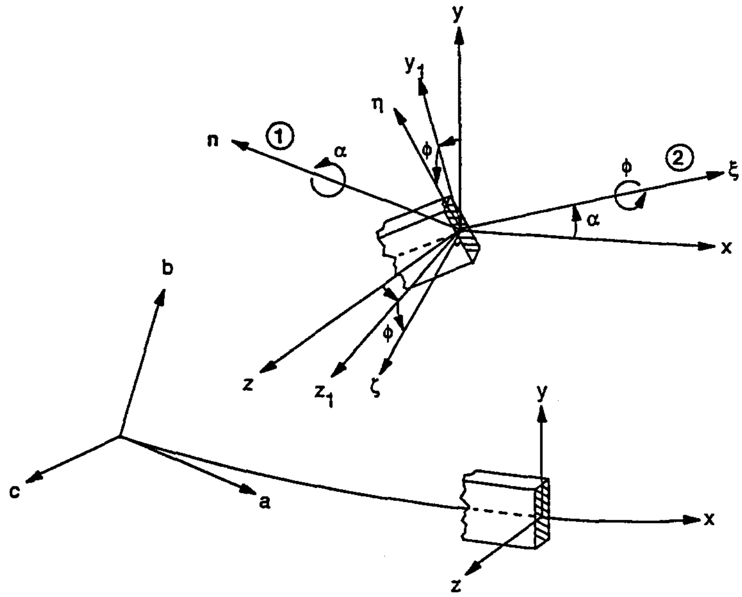

Fig. 3. Two successive Euler angle rotations of a differential beam element.

$$\{\mathbf{i}_{1\hat{2}\hat{3}}\} = [B(\alpha)]\{\mathbf{i}_{xyz}\}, \tag{8}$$

where $\mathbf{i}_{\hat{2}}$ and $\mathbf{i}_{\hat{3}}$ are unit vectors along the y_1 and z_1 axes, respectively.

Next we need to represent $[B(\alpha)]$ in terms of the displacements u, v and w. In Fig. 4, we show the relationship between the reference line and the Euler angles α and ϕ. It follows from Fig. 4 and eqns (6a) and (6b) that the displacement vectors of the two reference points (at s) and q (at $s+\mathrm{d}s$) are

$$p: \quad \mathbf{D}_1 = u\mathbf{i}_x + v\mathbf{i}_y + w\mathbf{i}_z$$

$$q: \quad \mathbf{D}_2 = \mathbf{D}_1 + \frac{\partial \mathbf{D}_1}{\partial s}\,\mathrm{d}s$$

$$= \mathbf{D}_1 + [(u' - vk_3 + wk_2)\mathbf{I}_x + (v' + uk_3 - wk_1)\mathbf{i}_y + (w' - uk_2 + vk_1)\mathbf{i}_z]\,\mathrm{d}s. \tag{9}$$

Hence, the vector from the deformed reference point $\hat{p}$ to the deformed reference point $\hat{q}$ is

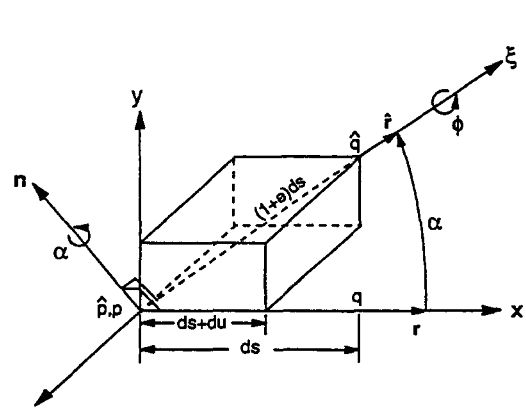

Fig. 4. Relationship between the reference line and the Euler angles.

$$\vec{pq} = \mathrm{d}s\mathbf{i}_x + \mathbf{D}_2 - \mathbf{D}_1$$
$$= [(1+u'-vk_3+wk_2)\mathbf{i}_x + (v'+uk_3-wk_1)\mathbf{i}_y + (w'-uk_2+vk_1)\mathbf{i}_z]\,\mathrm{d}s. \tag{10}$$

Therefore

$$\mathbf{i}_1 = \frac{\vec{pq}}{(1+e)\,\mathrm{d}s} = T_{11}\mathbf{i}_x + T_{12}\mathbf{i}_y + T_{13}\mathbf{i}_z, \tag{11a}$$

where e is the axial strain along the reference line and

$$T_{11} = \frac{1+u'-vk_3+wk_2}{1+e}, \quad T_{12} = \frac{v'+uk_3-wk_1}{1+e}, \quad T_{13} = \frac{w'-uk_2+vk_1}{1+e}. \tag{11b}$$

It follows from eqn (10) and Fig. 4 that the relationship between the axial strain e and the displacements is

$$e = \frac{\overline{\hat{p}\hat{q}} - \mathrm{d}s}{\mathrm{d}s}$$
$$= \sqrt{(1+u'-vk_3+wk_2)^2 + (v'+uk_3-wk_1)^2 + (w'-uk_2+vk_1)^2} - 1. \tag{12}$$

A rotation axis n and a rotation angle α about the n axis are used to define the bending rotation. As shown in Fig. 4, we choose the axis n to be

$$\mathbf{n} \equiv \frac{\mathbf{i}_x \times \mathbf{i}_1}{|\mathbf{i}_x \times \mathbf{i}_1|} = n_1\mathbf{i}_x + n_2\mathbf{i}_y + n_3\mathbf{i}_z. \tag{13a}$$

Substituting for $\mathbf{i}_1$ from eqn (11a) into (13a), we obtain

$$n_1 = 0, \quad n_2 = \frac{-T_{13}}{\sqrt{T_{12}^2 + T_{13}^2}}, \quad n_3 = \frac{T_{12}}{\sqrt{T_{12}^2 + T_{13}^2}}. \tag{13b}$$

When the xyz coordinate system is rotated by an angle α with respect to the axis n, the rotated vector $\hat{\mathbf{r}}$ of an arbitrary vector $\mathbf{r}$, which is fixed on the xyz frame, is related to $\mathbf{r}$ and α by (Smith, 1976)

$$\hat{\mathbf{r}} = (1-\cos\alpha)(\mathbf{r}\cdot\mathbf{n})\mathbf{n} + \cos\alpha\,\mathbf{r} + \sin\alpha\,\mathbf{n}\times\mathbf{r}. \tag{14}$$

By letting $\mathbf{r} = \mathbf{i}_x$ and $\hat{\mathbf{r}} = \mathbf{i}_1$, $\mathbf{r} = \mathbf{i}_y$ and $\hat{\mathbf{r}} = \mathbf{i}_2$, and $\mathbf{r} = \mathbf{i}_z$ and $\hat{\mathbf{r}} = \mathbf{i}_3$ in eqn (14) individually and using eqn (8), we obtain

$$[B(\alpha)] = \begin{bmatrix} n_1^2 & n_1n_2 & n_1n_3 \\ & n_2^2 & n_2n_3 \\ \text{sym.} & & n_3^2 \end{bmatrix}(1-\cos\alpha) + [I]\cos\alpha + [P(n_1,n_2,n_3)]\sin\alpha, \tag{15}$$

where $[I]$ is the identity matrix. Substituting eqn (13b) into eqn (15), assuming that $0 \leqslant \alpha < 180°$, and using the relationships

$$T_{11}^2 + T_{12}^2 + T_{13}^2 = 1, \quad \cos\alpha = \mathbf{i}_1\cdot\mathbf{i}_x = T_{11}, \quad \sin\alpha = |\mathbf{i}_1\times\mathbf{i}_x| = \sqrt{T_{12}^2 + T_{13}^2}, \tag{16}$$

we obtain

$$[B(\alpha)] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ -T_{12} & T_{11}+T_{13}^2/(1+T_{11}) & -T_{12}T_{13}/(1+T_{11}) \\ -T_{13} & -T_{12}T_{13}/(1+T_{11}) & T_{11}+T_{12}^2/(1+T_{11}) \end{bmatrix}. \tag{17a}$$

We note from eqn (17a) that $[B(\alpha)]$ is indeterminate only when $T_{11} = -1$, which corresponds to $\alpha = 180°$ and $T_{12} = T_{13} = 0$ according to eqn (16). Hence, it follows from eqn (13b) that $n_1 = 0$ and n_2 and n_3 are indeterminate. In this case, eqn (15) reduces to

$$[B(\alpha)] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2n_2^2-1 & 2n_2n_3 \\ 0 & 2n_2n_3 & 2n_3^2-1 \end{bmatrix} \quad \text{for } \alpha = 180°. \tag{17b}$$

Using the concept of continuity, one can determine the values of n_2, n_3 and $[B(\alpha)]$ at a particular point $s = s_p$ by comparing eqn (17b) with eqn (17a) at adjacent points $s = s_p^-$ or s_p^+. Hence, any arbitrary deformation (i.e. $0 \leqslant \alpha \leqslant 180°$) can be modeled.

To complete the geometry analysis we need expressions for the curvatures. Using eqns (7a) and (6a) and the identity $[T]^{-1} = [T]^T$ (because $[T]$ is a unitary matrix), we obtain

$$\frac{\partial}{\partial s}\{\mathbf{i}_{123}\} = [\mathrm{K}]\{\mathbf{i}_{123}\}, \tag{18a}$$

where the deformed curvature matrix [K] is given by

$$[\mathrm{K}] \equiv \frac{\partial [T]}{\partial s}[T]^T + [T][k][T]^T = [P(\rho_1, \rho_2, \rho_3)]. \tag{18b}$$

The twisting curvature ρ_1 and the bending curvatures ρ_2 and ρ_3 can be obtained by using eqns (18a) and (18b) and the orthonormality property of $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ as

$$\begin{aligned} \rho_1 &\equiv \frac{\partial \mathbf{i}_2}{\partial s}\cdot\mathbf{i}_3 = \sum_{i=1}^{3}(T'_{2i}T_{3i} + T_{1i}k_i) \\ \rho_2 &\equiv -\frac{\partial \mathbf{i}_1}{\partial s}\cdot\mathbf{i}_3 = \sum_{i=1}^{3}(-T'_{1i}T_{3i} + T_{2i}k_i) \\ \rho_3 &\equiv \frac{\partial \mathbf{i}_1}{\partial s}\cdot\mathbf{i}_2 = \sum_{i=1}^{3}(T'_{1i}T_{2i} + T_{3i}k_i). \end{aligned} \tag{19}$$

We note that, when there are no elastic deformations, $[T]$ is an identity matrix and $[\mathrm{K}] = [k]$.

Using eqns (12), (11b), and (17a), one can express the elements of the transformation matrix $[T]$ defined in eqn (7b) and the curvatures defined in eqn (19) in terms of u, v, w and ϕ and their derivatives. We note that the ρ_i are not real curvatures because the differentiation is with respect to the undeformed element length ds and not the deformed element length $(1+e)$ ds.

3. THREE-DIMENSIONAL STRAIN FIELD

To fully account for the change in configuration, we use local engineering stress and strain measures. The movement of the observed cross-section consists of two parts. The first part is a rigid-body motion (see Fig. 5), which is due to the rigid-body translation $U(t)\mathbf{i}_a + V(t)\mathbf{i}_b + W(t)\mathbf{i}_c$ and rotation $\Omega\mathbf{i}_h$ of the rotor hub, the displacements u, v and w of the reference point, and the rotation angle ϕ. This rigid-body motion rotates the dy and dz sides of the observed cross-section so that they are parallel to the η and ζ axes, respectively. The second part is a local, strainable displacement vector $\mathbf{U}$, which results in strains. Because the rigid-body motion does not result in any strain energy, to calculate the elastic energy we only need to deal with the strainable, local displacement field **U**. To account for three-dimensional stress effects, we include both in-plane and out-of-plane deformations in the displacement field. Thus, we represent the local displacement field as

$$\begin{aligned} u_1(s,y,z,t) &= u_1^0(s,t)+z[\theta_2(s,t)-\theta_{20}(s)]-y[\theta_3(s,t)-\theta_{30}(s)] \\ &\quad +z\gamma_5(s,t)+y\gamma_6(s,t)+W_1(s,y,z,t) \\ u_2(s,y,z,t) &= u_2^0(s,t)-z[\theta_1(s,t)-\theta_{10}(s)]+W_2(s,y,z,t) \\ u_3(s,y,z,t) &= u_3^0(s,t)+y[\theta_1(s,t)-\theta_{10}(s)]+W_3(s,y,z,t). \end{aligned} \tag{20}$$

Here, the Lagrangian coordinates s, y and z are used to express all functions because engineering strains are referred to the undeformed length. Moreover, u_1, u_2 and u_3 are local, strainable displacements with respect to the ξ, η and ζ axes, respectively; $u_i^0(s,\ t) \equiv u_i(s,0,0,t)$, $i = 1,2,3$; θ_1, θ_2 and θ_3 are the rotation angles of the observed cross-section with respect to the ξ, η and ζ axes, respectively; θ_{10}, θ_{20} and θ_{30} are the initial rotation angles of the observed cross-section with respect to the ξ, η and ζ axes, respectively; γ_5 and γ_6 are the out-of-plane shear rotation angles evaluated at the reference axis, that is, $\gamma_5 \equiv u_{1z}|_{y=z=0}$, $\gamma_6 \equiv u_{1y}|_{y=z=0}$. The function $W_1(s,y,z,t)$ represents the out-of-plane warping due to higher order transverse shear deformations (first-order shear deformations are accounted for by $z\gamma_5$ and $y\gamma_6$) and torsional warping. The functions $W_2(s,y,z,t)$ and $W_3(s,y,z,t)$ represent the in-plane warpings due to the Poisson effect and the action of extension and/or bendings.

Because the $\xi\eta\zeta$ is a local coordinate system attached to the observed cross-section and the unit vector $\mathbf{i}_1$ is tangent to the deformed reference axis, we have

$$u_1^0 = u_2^0 = u_3^0 = \theta_{10} = \theta_{20} = \theta_{30} = \theta_1 = \theta_2 = \theta_3 = \frac{\partial u_2^0}{\partial s} = \frac{\partial u_3^0}{\partial s} = 0. \tag{21}$$

Because the local displacement vector **U** is given by

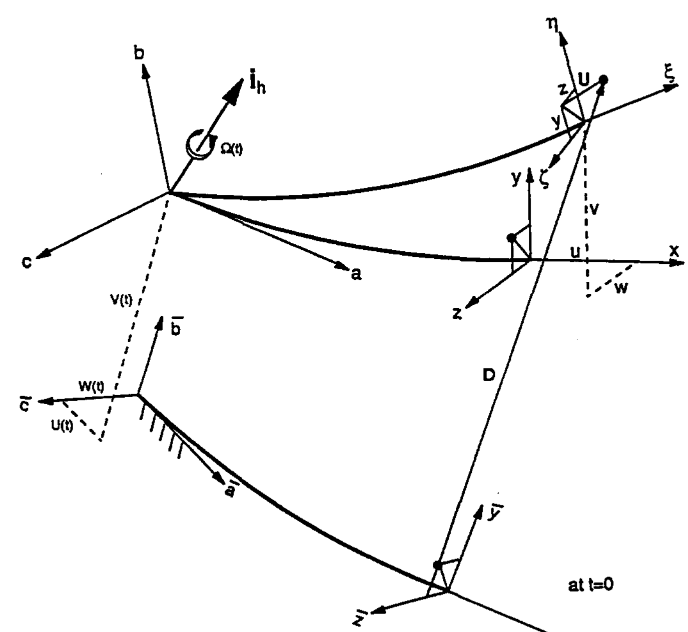

Fig. 5. Displacements and relations among coordinate systems.

$$\mathbf{U} = u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3 \tag{22}$$

we obtain from eqns (22), (20), (18a), (18b) and (21) that

$$\begin{aligned}\frac{\partial \mathbf{U}}{\partial s} &= \frac{\partial u_1}{\partial s}\mathbf{i}_1 + \frac{\partial u_2}{\partial s}\mathbf{i}_2 + \frac{\partial u_3}{\partial s}\mathbf{i}_3 + u_1\frac{\partial \mathbf{i}_1}{\partial s} + u_2\frac{\partial \mathbf{i}_2}{\partial s} + u_3\frac{\partial \mathbf{i}_3}{\partial s} \\ &= [e + z(\rho_2 - k_2) - y(\rho_3 - k_3) + z\gamma_5' + y\gamma_6' + W_1']\mathbf{i}_1 + [-z(\rho_1 - k_1) + W_2']\mathbf{i}_2 \\ &\quad + [y(\rho_1 - k_1) + W_3']\mathbf{i}_3 + (W_3\rho_2 - W_2\rho_3)\mathbf{i}_1 \\ &\quad + [(W_1 + z\gamma_5 + y\gamma_6)\rho_3 - W_3\rho_1]\mathbf{i}_2 + [W_2\rho_1 - (W_1 + z\gamma_5 + y\gamma_6)\rho_2]\mathbf{i}_3 \end{aligned} \tag{23a}$$

$$\frac{\partial \mathbf{U}}{\partial y} = \frac{\partial u_1}{\partial y}\mathbf{i}_1 + \frac{\partial u_2}{\partial y}\mathbf{i}_2 + \frac{\partial u_3}{\partial y}\mathbf{i}_3 = (W_{1y} + \gamma_6)\mathbf{i}_1 + W_{2y}\mathbf{i}_2 + W_{3y}\mathbf{i}_3 \tag{23b}$$

$$\frac{\partial \mathbf{U}}{\partial z} = \frac{\partial u_1}{\partial z}\mathbf{i}_1 + \frac{\partial u_2}{\partial z}\mathbf{i}_2 + \frac{\partial u_3}{\partial z}\mathbf{i}_3 = (W_{1z} + \gamma_5)\mathbf{i}_1 + W_{2z}\mathbf{i}_2 + W_{3z}\mathbf{i}_3, \tag{23c}$$

where $W_{iy} \equiv \partial W_i/\partial y$, $W_{iz} \equiv \partial W_i/\partial z$, and

$$e \equiv \frac{\partial u_1^0}{\partial s}, \quad \rho_i \equiv \frac{\partial \theta_i}{\partial s}, \quad k_i \equiv \frac{\partial \theta_{i0}}{\partial s}, \quad i = 1, 2, 3. \tag{24}$$

The strains are obtained as

$$\begin{aligned}\varepsilon_{11} &= \frac{\partial \mathbf{U}}{\partial s}\cdot\mathbf{i}_1 = e + z(\rho_2 - k_2) - y(\rho_3 - k_3) + z\gamma_5' + y\gamma_6' + W_1' + W_3\rho_2 - W_2\rho_3 \\ \varepsilon_{12} &= \frac{\partial \mathbf{U}}{\partial y}\cdot\mathbf{i}_1 + \frac{\partial \mathbf{U}}{\partial s}\cdot\mathbf{i}_2 = -z(\rho_1 - k_1) + W_2' + W_{1y} + \gamma_6 + (W_1 + z\gamma_5 + y\gamma_6)\rho_3 - W_3\rho_1 \\ \varepsilon_{13} &= \frac{\partial U}{\partial z}\cdot\mathbf{i}_1 + \frac{\partial \mathbf{U}}{\partial s}\cdot\mathbf{i}_3 = y(\rho_1 - k_1) + W_3' + W_{1z} + \gamma_5 + W_2\rho_1 - (W_1 + z\gamma_5 + y\gamma_6)\rho_2 \\ \varepsilon_{22} &= \frac{\partial \mathbf{U}}{\partial y}\cdot\mathbf{i}_2 = W_{2y} \\ \varepsilon_{23} &= \frac{\partial \mathbf{U}}{\partial y}\cdot\mathbf{i}_3 + \frac{\partial \mathbf{U}}{\partial z}\cdot\mathbf{i}_2 = W_{3y} + W_{2z} \\ \varepsilon_{33} &= \frac{\partial \mathbf{U}}{\partial z}\cdot\mathbf{i}_3 = W_{3z}. \end{aligned} \tag{25}$$

In Appendix A, we show that the ε_{ij} are local engineering strains defined with respect to the deformed coordinate system $\xi\eta\zeta$. One can rewrite the strains in compact matrix form as

$$\{\varepsilon\} = [X]\{\psi\} + [\hat{I}]\{W'\} + [\partial]\{W\} - [\hat{\mathbf{K}}]\{W\} + [\bar{X}]\{\psi\} + \{\bar{\psi}\}, \tag{26a}$$

where

$$\{\varepsilon\} \equiv \{\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{22}, \varepsilon_{23}, \varepsilon_{33}\}^T$$
$$\{\psi\} \equiv \{e, \gamma_6, \gamma_5, \rho_1 - k_1, \rho_2 - k_2, \rho_3 - k_3\}^T$$
$$\{\bar{\psi}\} \equiv \{y\gamma_6' + z\gamma_5', 0, 0, 0, 0, 0\}^T$$
$$\{W\} = \{W_1, W_2, W_3\}^T$$
$$[X] \equiv \begin{bmatrix} [I] & [P(0,y,z)] \\ [0] & [0] \end{bmatrix}, \quad [\hat{I}] \equiv \begin{bmatrix} [I] \\ [0] \end{bmatrix}, \quad [\hat{K}] \equiv \begin{bmatrix} [K] \\ [0] \end{bmatrix}$$
$$[\partial] \equiv \begin{bmatrix} 0 & 0 & 0 \\ \partial/\partial y & 0 & 0 \\ \partial/\partial z & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ 0 & 0 & \partial/\partial z \end{bmatrix}, \quad [\bar{X}] \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_3 y & \rho_3 z & 0 & 0 & 0 \\ 0 & -\rho_2 y & -\rho_2 z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{26b}$$

and [0] is a 3×3 null matrix. We note that $[\hat{K}]$ and $[\bar{X}]$ are functions of the deformed curvatures ρ_i.

Using a finite element discretization scheme (Giavotto *et al.*, 1983; Pai and Nayfeh, 1993), we have

$$\{W\} = [N(y,z)]\{\hat{\Gamma}(s,t)\}, \tag{27a}$$

where $\{\hat{\Gamma}\}$ is a $3n \times 1$ vector consisting of the three relative displacements (with respect to the ξ, η and ζ axes) of each node on the cross-section, n is the total number of nodes on the cross-section, and $[N]$ is a $3 \times 3n$ matrix of appropriate two-dimensional finite element interpolation functions. A two-dimensional, linear, static, sectional analysis gives the central solutions (Pai and Nayfeh, 1993)

$$\{\hat{\Gamma}\} = [G]\{\tau\}, \quad \{\hat{\Gamma}'\} = [G']\{\tau\}, \quad \{\psi\} = [Y]\{\tau\}, \tag{27b}$$

where $[G]$ and $[G']$ are $3n \times 6$ constant matrices, $[Y]$ is a 6×6 constant matrix, and $\{\tau\}$ is a vector of stress resultants defined as

$$\{\tau\} \equiv \{F_1, F_2, F_3, M_1, M_2, M_3\}^T$$
$$\equiv \int_A \{\sigma_{11}, \sigma_{12}, \sigma_{13}, y\sigma_{13} - z\sigma_{12}, z\sigma_{11}, -y\sigma_{11}\}^T \, dy \, dz. \tag{27c}$$

It follows from eqns (27a), (27b) and (26a) that

$$\{W\} = [N][G][Y]^{-1}\{\psi\}, \quad \{W'\} = [N][G'][Y]^{-1}\{\psi\} \tag{28a}$$
$$\{\varepsilon\} = [S]\{\psi\} + \{\bar{\psi}\}$$
$$[S] \equiv [X] + [\hat{I}][N][G'][Y]^{-1} + [\partial][N][G][Y]^{-1} - [\hat{K}][N][G][Y]^{-1} + [\bar{X}], \tag{28b}$$

where $[S]$ is a 6×6 matrix.

Next, we need constitutive equations to relate the force strains (i.e. e, γ_5 and γ_6) and the moment strains (i.e. curvatures ρ_i) to the stresses. For a general anisotropic beam (e.g. woven composite beams, built-up beams), the stiffness matrix $[\tilde{Q}]$ is a full symmetric matrix because there are elastic couplings among all the deformations. Hence, a general stress–strain relation is represented by

$$\{\sigma\} = [\tilde{Q}]\{\varepsilon\}, \quad \{\sigma\} \equiv \{\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33}\}^T, \tag{29}$$

where the σ_{ij} are local engineering stresses defined with respect to the deformed coordinate system $\xi\eta\zeta$. For notational convenience, $\{\sigma\}$ and $\{\varepsilon\}$ are often written as

$$\{\sigma\} = \{\sigma_1, \sigma_6, \sigma_5, \sigma_2, \sigma_4, \sigma_3\}^T \quad \text{and} \quad \{\varepsilon\} = \{\varepsilon_1, \varepsilon_6, \varepsilon_5, \varepsilon_2, \varepsilon_4, \varepsilon_3\}^T.$$

4. DERIVATION OF THE EQUATIONS OF MOTION

The extended Hamilton principle is used to derive the equations of motion, which can be stated as

$$0 = \int_0^t (\delta T - \delta V + \delta W_{nc})\, \mathrm{d}t \tag{30}$$

where variations of the kinetic energy T and the elastic potential energy V are given by

$$\delta T = -\int_0^L \int_A \rho \ddot{\mathbf{D}} \cdot \delta \mathbf{D}\, \mathrm{d}A\, \mathrm{d}s \tag{31a}$$

$$\begin{aligned}
\delta V = \int_0^L \int_A \Bigg\{ & \left(\sigma_{11}\delta\frac{\partial \mathbf{U}}{\partial s}\mathrm{d}s\cdot\mathbf{i}_1 + \sigma_{12}\delta\frac{\partial \mathbf{U}}{\partial s}\mathrm{d}s\cdot\mathbf{i}_2 + \sigma_{13}\delta\frac{\partial \mathbf{U}}{\partial s}\mathrm{d}s\cdot\mathbf{i}_3\right)\mathrm{d}y\,\mathrm{d}z \\
& + \left(\sigma_{21}\delta\frac{\partial \mathbf{U}}{\partial y}\mathrm{d}y\cdot\mathbf{i}_1 + \sigma_{22}\delta\frac{\partial \mathbf{U}}{\partial y}\mathrm{d}y\cdot\mathbf{i}_2 + \sigma_{23}\delta\frac{\partial \mathbf{U}}{\partial y}\mathrm{d}y\cdot\mathbf{i}_3\right)\mathrm{d}s\,\mathrm{d}z \\
& + \left(\sigma_{31}\delta\frac{\partial \mathbf{U}}{\partial z}\mathrm{d}z\cdot\mathbf{i}_1 + \sigma_{32}\delta\frac{\partial \mathbf{U}}{\partial z}\mathrm{d}z\cdot\mathbf{i}_2 + \sigma_{33}\delta\frac{\partial \mathbf{U}}{\partial z}\mathrm{d}z\cdot\mathbf{i}_3\right)\mathrm{d}s\,\mathrm{d}y \Bigg\} \\
= \int_0^L \int_A & (\sigma_{11}\delta\varepsilon_{11} + \sigma_{12}\delta\varepsilon_{12} + \sigma_{13}\delta\varepsilon_{13} + \sigma_{22}\delta\varepsilon_{22} + \sigma_{23}\delta\varepsilon_{23} + \sigma_{33}\delta\varepsilon_{33})\, \mathrm{d}A\, \mathrm{d}s
\end{aligned} \tag{31b}$$

and δW_{nc} denotes variation of the non-conservative energy W_{nc}, which is problem-dependent and will not be considered in the derivation. Here, A denotes the undeformed cross-sectional area, L is the total undeformed arc length of the beam, ρ is the mass density of the material, the point denotes an inner product of vectors, $\mathbf{D}$ is the absolute displacement vector of an arbitrary point on the observed cross-section (see Fig. 5), and $\ddot{\mathbf{D}} \equiv \mathrm{d}^2\mathbf{D}/\mathrm{d}t^2$. The local engineering stresses σ_{ij} are evaluated with respect to the undeformed area and pointing along the direction of $\mathbf{i}_j$, and the local engineering strains ε_{ij} are defined in eqn (25).

4.1. *Inertial terms*

It follows from Fig. 5 that the absolute displacement vector $\mathbf{D}$ (i.e. displacements with respect to an inertial coordinate system) of a generic point on the observed cross-section is given by

$$\begin{aligned}
\mathbf{D} = U(t)\mathbf{i}_{\bar{a}} + V(t)\mathbf{i}_{\bar{b}} + W(t)\mathbf{i}_{\bar{c}} + A\mathbf{i}_a + B\mathbf{i}_b + C\mathbf{i}_c + y\mathbf{i}_y + z\mathbf{i}_z - A\mathbf{i}_{\bar{a}} - B\mathbf{i}_{\bar{b}} - C\mathbf{i}_{\bar{c}} - y\mathbf{i}_{\bar{y}} - z\mathbf{i}_{\bar{z}} \\
+ u\mathbf{i}_x + v\mathbf{i}_y + w\mathbf{i}_z + y\mathbf{i}_2 + z\mathbf{i}_3 - y\mathbf{i}_y - z\mathbf{i}_z + u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3.
\end{aligned} \tag{32a}$$

Although the in-plane warping displacements W_2 and W_3 may influence the elastic properties, their contribution to the inertial forces is insignificant. Neglecting W_2 and W_3 in eqn (20) and using eqns (21) and (28a), we obtain

$$\begin{aligned}
u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3 &= \bar{W}_1\mathbf{i}_1 \\
\bar{W}_1 \equiv z\gamma_5 + y\gamma_6 + W_1 &= z\gamma_5 + y\gamma_6 + \{1,0,0\}[N][G][Y]^{-1}\{\psi\},
\end{aligned} \tag{32b}$$

where $\bar{W}_1$ is the total out-of-plane warping. Neglecting the influence of extension and bendings (i.e. e, ρ_2-k_2, ρ_3-k_3) on the out-of-plane warping W_1, we rewrite eqn (32b) as

$$u_1\mathbf{i}_1+u_2\mathbf{i}_2+u_3\mathbf{i}_3 = [g_1(\rho_1-k_1)+g_2\gamma_5+g_3\gamma_6]\mathbf{i}_1, \tag{32c}$$

where

$$\{g_1,g_2,g_3\} \equiv \{1,0,0\}[N][G][Y]^{-1}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\\1&0&0\\0&0&0\\0&0&0\end{bmatrix}+\{0,z,y\}. \tag{32d}$$

Here g_1 is the torsional warping function and g_2 and g_3 are shear warping functions.

To derive the equations governing u, v, w, γ_5, γ_6 and ϕ by using a variational method, one needs to determine the variations of these dependent variables. However, because the unit vectors $\mathbf{i}_k$ along the ξ, η and ζ axes are functions of u, v, w and ϕ, as shown in eqns (7), (17a) and (11b), we need to express $\delta\mathbf{i}_1$, $\delta\mathbf{i}_2$ and $\delta\mathbf{i}_3$ in terms of δu, δv, δw and $\delta\phi$. Since $\delta\mathbf{i}_1$, $\delta\mathbf{i}_2$ and $\delta\mathbf{i}_3$ are due to virtual rigid-body rotations of the observed cross-section, we have

$$\delta\{\mathbf{i}_{123}\} = [P(\delta\theta_1,\delta\theta_2,\delta\theta_3)]\{\mathbf{i}_{123}\}, \tag{33}$$

where $\delta\theta_1$, $\delta\theta_2$ and $\delta\theta_3$ are virtual rotations with respect to the ξ, η and ζ axes, respectively.

It follows from eqns (32a), (32c) and (33) that

$$\begin{aligned}\delta\mathbf{D} &= \mathbf{i}_x\delta u+\mathbf{i}_y\delta v+\mathbf{i}_z\delta w+g\delta\mathbf{i}_1+y\delta\mathbf{i}_2+z\delta\mathbf{i}_3+\mathbf{i}_1\delta g\\ &= \{\mathbf{i}_{xyz}\}^T\begin{Bmatrix}\delta u\\ \delta v\\ \delta w\end{Bmatrix}+\{\mathbf{i}_{123}\}^T[P(g,y,z)]\begin{Bmatrix}\delta\theta_1\\ \delta\theta_2\\ \delta\theta_3\end{Bmatrix}+\mathbf{i}_1(g_1\delta\rho_1+g_2\delta\gamma_5+g_3\delta\gamma_6),\end{aligned} \tag{34}$$

where

$$g \equiv g_1\bar{\rho}_1+g_2\gamma_5+g_3\gamma_6, \quad \bar{\rho}_1 \equiv \rho_1-k_1. \tag{35}$$

To determine $\ddot{\mathbf{D}}$, we need the angular velocities and rigid-body transformation matrices of the three coordinate systems. We define the angular velocity $\vec{\omega}$ of the $\xi\eta\zeta$ frame as

$$\vec{\omega} = \omega_1\mathbf{i}_1+\omega_2\mathbf{i}_2+\omega_3\mathbf{i}_3 \tag{36a}$$

the angular velocity $\vec{\omega}^x$ of the xyz frame as

$$\vec{\omega}^x = \omega_1^x\mathbf{i}_x+\omega_2^x\mathbf{i}_y+\omega_3^x\mathbf{i}_z \tag{36b}$$

and the angular velocity $\vec{\omega}^a$ of the abc frame as

$$\vec{\omega}^a = \omega_1^a\mathbf{i}_a+\omega_2^a\mathbf{i}_b+\omega_3^a\mathbf{i}_c. \tag{36c}$$

We note that

$$\Omega\mathbf{i}_h = \vec{\omega}^a = \vec{\omega}^x. \tag{36d}$$

The unit vectors of the $\bar{a}\bar{b}\bar{c}$ and abc systems are related by

$$\{\mathbf{i}_{abc}\} = [T^a]\{\mathbf{i}_{\bar{a}\bar{b}\bar{c}}\}, \tag{37}$$

where $[T^a]$ is due to rigid-body rotations, as shown in Fig. 5. We assume that rigid-body motions of the reference frame abc are known and hence $[T^a]$ is a function of three known rotations $\Theta_1(t)$, $\Theta_2(t)$ and $\Theta_3(t)$. For example, if these three angles are sequential Euler angles (first, Θ_1 around the $\bar{a}$ axis; second, Θ_2 around the rotated $\bar{b}$ axis; lastly, Θ_3 around the rotated $\bar{c}$ axis), then

$$[T^a] = \begin{bmatrix} \cos\Theta_3 & \sin\Theta_3 & 0 \\ -\sin\Theta_3 & \cos\Theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta_2 & 0 & -\sin\Theta_2 \\ 0 & 1 & 0 \\ \sin\Theta_2 & 0 & \cos\Theta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta_1 & \sin\Theta_1 \\ 0 & -\sin\Theta_1 & \cos\Theta_1 \end{bmatrix}. \tag{38}$$

We note that if the governing equations of the rigid-body motions of the reference frame abc (e.g. fuselage) are to be determined as part of the solution, then δU, δV, δW, $\delta\Theta_1$, $\delta\Theta_2$ and $\delta\Theta_3$ are nontrivial.

Taking the time derivative of eqn (37) yields

$$\frac{\partial}{\partial t}\{\mathbf{i}_{abc}\} = [\dot{T}^a][T^a]^T\{\mathbf{i}_{abc}\}. \tag{39a}$$

Also, it follows from eqns (1f) and (36c) that

$$\frac{\partial}{\partial t}\{\mathbf{i}_{abc}\} = \vec{\omega}^a \times \{\mathbf{i}_{abc}\} = [P(\omega^a)]\{\mathbf{i}_{abc}\}. \tag{39b}$$

Hence, we obtain from eqns (39a) and (39b) that

$$\omega_1^a = \sum_{i=1}^{3} \dot{T}_{2i}^a T_{3i}^a, \quad \omega_2^a = \sum_{i=1}^{3} \dot{T}_{3i}^a T_{1i}^a, \quad \omega_3^a = \sum_{i=1}^{3} \dot{T}_{1i}^a T_{2i}^a. \tag{40}$$

Moreover, it follows from eqns (36d), (36b), (36c) and (5a) that

$$\omega_i^x = \sum_{j=1}^{3} \omega_j^a T_{ij}^x, \quad i = 1, 2, 3. \tag{41}$$

It follows from eqns (1f) and (36a) that

$$\frac{\partial}{\partial t}\{\mathbf{i}_{123}\} = \vec{\omega} \times \{\mathbf{i}_{123}\} = [P(\omega)]\{\mathbf{i}_{123}\}. \tag{42a}$$

Also it follows from eqn (7a) that

$$\frac{\partial}{\partial t}\{\mathbf{i}_{123}\} = ([\dot{T}][T]^T + [T][P(\omega^x)][T]^T)\{\mathbf{i}_{123}\}. \tag{42b}$$

Using eqns (42a) and (42b) and the orthonormality of the $\mathbf{i}_j$, we obtain

$$\omega_1 = \sum_{j=1}^{3} (\dot{T}_{2j}T_{3j} + \omega_j^x T_{1j}), \quad \omega_2 = \sum_{j=1}^{3} (\dot{T}_{3j}T_{1j} + \omega_j^x T_{2j}), \quad \omega_3 = \sum_{j=1}^{3} (\dot{T}_{1j}T_{2j} + \omega_j^x T_{3j}). \tag{43}$$

Equations (40) and (41) express the components of $\vec{\omega}^a$ and $\vec{\omega}^x$ in terms of the Θ_i, and eqns (43) express the components of $\vec{\omega}$ in terms of Θ_i, u, v, w, ϕ, and their derivatives.

Using eqns (32a), (32c) and (1c–f), we obtain

$$\ddot{\mathbf{D}} = \ddot{U}\mathbf{i}_{\bar{a}} + \ddot{V}\mathbf{i}_{\bar{b}} + \ddot{W}\mathbf{i}_{\bar{c}} + A\ddot{\mathbf{i}}_a + B\ddot{\mathbf{i}}_b + C\ddot{\mathbf{i}}_c + \ddot{u}\mathbf{i}_x + \ddot{v}\mathbf{i}_y + \ddot{w}\mathbf{i}_z$$
$$+ 2(\dot{u}\dot{\mathbf{i}}_x + \dot{v}\dot{\mathbf{i}}_y + \dot{w}\dot{\mathbf{i}}_z) + u\ddot{\mathbf{i}}_x + v\ddot{\mathbf{i}}_y + w\ddot{\mathbf{i}}_z + g\ddot{\mathbf{i}}_1 + y\ddot{\mathbf{i}}_2 + z\ddot{\mathbf{i}}_3 + \ddot{g}\mathbf{i}_1 + 2\dot{g}\dot{\mathbf{i}}_1$$
$$= \{\ddot{U}, \ddot{V}, \ddot{W}\}\{\mathbf{i}_{\bar{a}\bar{b}\bar{c}}\} - (\{\dot{\omega}^a\}^T[P(A,B,C)] + \{\omega^a\}^T[P(A,B,C)][P(\omega^a)])\{\mathbf{i}_{abc}\}$$
$$+ (\{\ddot{u},\ddot{v},\ddot{w}\} - \{\dot{\omega}^x\}^T[P(u,v,w)] - \{\omega^x\}^T[P(u,v,w)][P(\omega^x)] - 2\{\omega^x\}^T[P(\dot{u},\dot{v},\dot{w})])\{\mathbf{i}_{xyz}\}$$
$$- (\{\dot{\omega}\}^T[P(g,y,z)] + \{\omega\}^T[P(g,y,z)][P(\omega)] - \{\ddot{g}, 2\dot{g}\omega_3, -2\dot{g}\omega_2\})\{\mathbf{i}_{123}\}. \quad (44)$$

Substituting eqns (34) and (44) into eqn (31a) and integrating the result yields

$$\delta T = -\int_0^L \left(\{A_u, A_v, A_w\} \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix} + \{A_{\theta_1}, A_{\theta_2}, A_{\theta_3}\} \begin{Bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{Bmatrix} + \{A_{\rho_1}, A_{\gamma_5}, A_{\gamma_6}\} \begin{Bmatrix} \delta\rho_1 \\ \delta\gamma_5 \\ \delta\gamma_6 \end{Bmatrix} \right) \mathrm{d}s, \quad (45)$$

where

$$\{A_u, A_v, A_w\} \equiv (\{Q_1\}^T[T^x]^T + \{Q_2\}^T)I_{11} - (\{\dot{\omega}\}^T[I_1] + \{\omega\}^T[I_1][P(\omega)] - \{f_{11}, f_{12}, f_{13}\})[T]$$
$$\{A_{\theta_1}, A_{\theta_2}, A_{\theta_3}\} \equiv (\{Q_1\}^T[T^x]^T + \{Q_2\}^T)[T]^T[I_1] + \{\dot{\omega}\}^T[I_2] + \{\omega\}^T[I_2][P(\omega)] + \{f_{21}, f_{22}, f_{23}\}$$
$$\{A_{\rho_1}, A_{\gamma_5}, A_{\gamma_6}\} \equiv (\{Q_1\}^T[T^x]^T + \{Q_2\}^T)[T]^T[I_3] + \{f_{51}, f_{52}, f_{53}\}$$
$$\{Q_1\} \equiv [T^a]\{\ddot{U}, \ddot{V}, \ddot{W}\}^T + [P(A,B,C)]\{\dot{\omega}^a\} - [P(\omega^a)][P(A,B,C)]\{\omega^a\}$$
$$\{Q_2\} \equiv \{\ddot{u}, \ddot{v}, \ddot{w}\}^T + [P(u,v,w)]\{\dot{\omega}^x\} - [P(\omega^x)][P(u,v,w)]\{\omega^x\} + 2[P(\dot{u},\dot{v},\dot{w})]\{\omega^x\}. \quad (46)$$

Here, we used the identity

$$\{\omega\}^T[P(g,y,z)]^T[P(\omega)][P(g,y,z)] = \{\omega\}^T[P(g,y,z)]^T[P(g,y,z)][P(\omega)] \quad (47)$$

which can be proved by direct expansion. The inertias are defined as

$$\begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} & I_{15} & I_{16} \\ & I_{22} & I_{23} & I_{24} & I_{25} & I_{26} \\ & & I_{33} & I_{34} & I_{35} & I_{36} \\ & & & I_{44} & I_{45} & I_{46} \\ & & & & I_{55} & I_{56} \\ & \text{sym.} & & & & I_{66} \end{bmatrix} \equiv \int_A \rho \begin{Bmatrix} 1 \\ y \\ z \\ g_1 \\ g_2 \\ g_3 \end{Bmatrix} \{1, y, z, g_1, g_2, g_3\}\, \mathrm{d}A \quad (48a)$$

$$[I_1] \equiv \int_A \rho[P(g,y,z)]\,\mathrm{d}A = \begin{bmatrix} 0 & I_{31} & -I_{21} \\ & 0 & I_{41}\bar{\rho}_1 + I_{51}\gamma_5 + I_{61}\gamma_6 \\ \text{skew-sym.} & & 0 \end{bmatrix} \quad (48b)$$

$$[I_2] \equiv \int_A \rho[P(g,y,z)]^T[P(g,y,z)]\,\mathrm{d}A$$
$$= \begin{bmatrix} I_{22}+I_{33} & -I_{24}\bar{\rho}_1 - I_{25}\gamma_5 - I_{26}\gamma_6 & -I_{34}\bar{\rho}_1 - I_{35}\gamma_5 - I_{36}\gamma_6 \\ & \bar{\rho}_1 f_{01} + \gamma_5 f_{02} + \gamma_6 f_{03} + I_{33} & -I_{23} \\ \text{sym.} & & \bar{\rho}_1 f_{01} + \gamma_5 f_{02} + \gamma_6 f_{03} + I_{22} \end{bmatrix} \quad (48c)$$

$$[I_3] \equiv \int_A \rho \begin{bmatrix} g_1 & g_2 & g_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{d}A = \begin{bmatrix} I_{14} & I_{15} & I_{16} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{48d}$$

where

$$\begin{Bmatrix} f_{01} \\ f_{02} \\ f_{03} \end{Bmatrix} \equiv \begin{Bmatrix} I_{44}\bar{\rho}_1 + I_{45}\gamma_5 + I_{46}\gamma_6 \\ I_{54}\bar{\rho}_1 + I_{55}\gamma_5 + I_{56}\gamma_6 \\ I_{64}\bar{\rho}_1 + I_{65}\gamma_5 + I_{66}\gamma_6 \end{Bmatrix}, \quad \begin{Bmatrix} f_{11} \\ f_{12} \\ f_{13} \end{Bmatrix} \equiv \begin{Bmatrix} I_{14}\ddot{\rho}_1 + I_{15}\ddot{\gamma}_5 + I_{16}\ddot{\gamma}_6 \\ 2\omega_3(I_{14}\dot{\rho}_1 + I_{15}\dot{\gamma}_5 + I_{16}\dot{\gamma}_6) \\ -2\omega_2(I_{14}\dot{\rho}_1 + I_{15}\dot{\gamma}_5 + I_{16}\dot{\gamma}_6) \end{Bmatrix}$$

$$\begin{Bmatrix} f_{21} \\ f_{22} \\ f_{23} \end{Bmatrix} \equiv \begin{Bmatrix} -2\omega_3(I_{34}\dot{\rho}_1 + I_{35}\dot{\gamma}_5 + I_{36}\dot{\gamma}_6) - 2\omega_2(I_{24}\dot{\rho}_1 + I_{25}\dot{\gamma}_5 + I_{26}\dot{\gamma}_6) \\ I_{34}\ddot{\rho}_1 + I_{35}\ddot{\gamma}_5 + I_{36}\ddot{\gamma}_6 + 2\omega_2(\dot{\rho}_1 f_{01} + \dot{\gamma}_5 f_{02} + \dot{\gamma}_6 f_{03}) \\ -I_{24}\ddot{\rho}_1 - I_{25}\ddot{\gamma}_5 - I_{26}\ddot{\gamma}_6 + 2\omega_3(\dot{\rho}_1 f_{01} + \dot{\gamma}_5 f_{02} + \dot{\gamma}_6 f_{03}) \end{Bmatrix}$$

$$\begin{Bmatrix} f_{31} \\ f_{32} \\ f_{33} \end{Bmatrix} \equiv \begin{Bmatrix} \dot{\omega}_2 I_{34} - \dot{\omega}_3 I_{24} \\ \dot{\omega}_2 I_{35} - \dot{\omega}_3 I_{25} \\ \dot{\omega}_2 I_{36} - \dot{\omega}_3 I_{26} \end{Bmatrix}, \quad \begin{Bmatrix} f_{41} \\ f_{42} \\ f_{43} \end{Bmatrix} \equiv \begin{Bmatrix} \omega_1\omega_3 I_{34} + \omega_1\omega_2 I_{24} - (\omega_2^2 + \omega_3^2) f_{01} \\ \omega_1\omega_3 I_{35} + \omega_1\omega_2 I_{25} - (\omega_2^2 + \omega_3^2) f_{02} \\ \omega_1\omega_3 I_{36} + \omega_1\omega_2 I_{26} - (\omega_2^2 + \omega_3^2) f_{03} \end{Bmatrix}$$

$$\begin{Bmatrix} f_{51} \\ f_{52} \\ f_{53} \end{Bmatrix} \equiv \begin{Bmatrix} f_{31} + f_{41} + \ddot{f}_{01} \\ f_{32} + f_{42} + \ddot{f}_{02} \\ f_{33} + f_{43} + \ddot{f}_{03} \end{Bmatrix}. \tag{49}$$

We note that the components of $\{Q_1\}$ represent the accelerations due to rigid-body translations and rotations of the reference frame *abc* and the components of $\{Q_2\}$ represent the accelerations due to the flexural displacements and the rotation of the hub. It follows from eqn (48a) that $I_{ij} = I_{ji}$ and $I_{12} = I_{13} = I_{23} = 0$ if ρ is constant, the x axis represents the centroidal line, and the y and z axes are principal axes of the cross-section.

4.2. *Structural terms*

Substituting eqns (29) and (28a, b) into eqn (31b), we obtain variation of the elastic energy as

$$\begin{aligned} \delta V = & \int_0^L \int_A (\{y\gamma_6' + z\gamma_5', 0, 0, 0, 0, 0\} + \{\psi\}^T[S]^T)[\tilde{Q}][S]\{\delta\psi\} \, \mathrm{d}A \, \mathrm{d}s \\ & + \int_0^L \int_A [\sigma_{11}(y\delta\gamma_6' + z\delta\gamma_5' + W_3\delta\rho_2 - W_2\delta\rho_3) + \sigma_{12}(\bar{W}_1\delta\rho_3 - W_3\delta\rho_1) \\ & + \sigma_{13}(W_2\delta\rho_1 - \bar{W}_1\delta\rho_2)] \, \mathrm{d}A \, \mathrm{d}s \\ = & \int_0^L [\{\hat{F}_1 + \bar{F}_1, \hat{F}_2 + \bar{F}_2, \hat{F}_3 + \bar{F}_3, \hat{M}_1 + \bar{M}_1, \hat{M}_2 + \bar{M}_2, \hat{M}_3 + \bar{M}_3\}\{\delta\psi\} \\ & - M_3\delta\gamma_6' + M_2\delta\gamma_5' + m_1\delta\rho_1 + m_2\delta\rho_2 + m_3\delta\rho_3] \, \mathrm{d}s, \end{aligned} \tag{50}$$

where the stress resultants and moments are defined as

$$\{\hat{F}_1, \hat{F}_2, \hat{F}_3, \hat{M}_1, \hat{M}_2, \hat{M}_3\}^T \equiv [\Phi]\{\psi\}$$

$$\{\bar{F}_1, \bar{F}_2, \bar{F}_3, \bar{M}_1, \bar{M}_2, \bar{M}_3\}^T \equiv [E]\{\gamma'\}$$

$$\begin{Bmatrix} -M_3 \\ M_2 \end{Bmatrix} \equiv \int_A \begin{Bmatrix} y \\ z \end{Bmatrix} \sigma_{11} \, \mathrm{d}A = [E]^T\{\psi\} + [F]\{\gamma'\} \tag{51a}$$

$$
\begin{aligned}
m_1 &\equiv \int_A (\sigma_{13}W_2 - \sigma_{12}W_3)\,\mathrm{d}A = \int_A \{W\}^T[[P(1,0,0)][0]]\{\sigma\}\,\mathrm{d}A \\
&= \{\psi\}^T[C^0]\{\psi\} + \{\psi\}^T[C^1]\{\gamma'\} \\
m_2 &\equiv \int_A (\sigma_{11}W_3 - \sigma_{13}\bar{W}_1)\,\mathrm{d}A = \int_A [\{W\}^T[[P(0,1,0)][0]]\{\sigma\} - \sigma_{13}(y\gamma_6 + z\gamma_5)]\,\mathrm{d}A \\
&= \{\psi\}^T[C^2]\{\psi\} + \{\psi\}^T[C^3]\{\gamma'\} - \{\psi\}^T[C^4]\{\gamma\} - \{\gamma'\}^T[C^5]\{\gamma\} \\
m_3 &\equiv \int_A (\sigma_{12}\bar{W}_1 - \sigma_{11}W_2)\,\mathrm{d}A = \int_A [\{W\}^T[[P(0,0,1)][0]]\{\sigma\} + \sigma_{12}(y\gamma_6 + z\gamma_5)]\,\mathrm{d}A \\
&= \{\psi\}^T[C^6]\{\psi\} + \{\psi\}^T[C^7]\{\gamma'\} + \{\psi\}^T[C^8]\{\gamma\} + \{\gamma'\}^T[C^9]\{\gamma\}.
\end{aligned} \tag{51b}
$$

Here, [0] is a 3×3 null matrix and

$$
\begin{aligned}
\{\gamma\} &\equiv \{\gamma_6, \gamma_5\}^T \\
\{\tilde{Q}_i\} &\equiv \{\tilde{Q}_{i1}, \tilde{Q}_{i6}, \tilde{Q}_{i5}, \tilde{Q}_{i2}, \tilde{Q}_{i4}, \tilde{Q}_{i3}\}^T \\
[\Phi] &\equiv \int_A [S]^T[\tilde{Q}][S]\,\mathrm{d}A = \begin{bmatrix} [A] & [B] \\ [B]^T & [D] \end{bmatrix} \\
[E] &\equiv \int_A [S]^T[y\{\tilde{Q}_1\}\ z\{\tilde{Q}_1\}]\,\mathrm{d}A = \begin{bmatrix} [E^1] \\ [E^2] \end{bmatrix} \\
[F] &\equiv \int_A \tilde{Q}_{11} \begin{bmatrix} y^2 & yz \\ yz & z^2 \end{bmatrix} \mathrm{d}A \\
[C^0] &\equiv [Y]^{-T}[G]^T \int_A [N]^T[[P(1,0,0)][0]][\tilde{Q}][S]\,\mathrm{d}A \\
[C^1] &\equiv [Y]^{-T}[G]^T \int_A [N]^T[[P(1,0,0)][0]][y\{\tilde{Q}_1\}\ z\{\tilde{Q}_1\}]\,\mathrm{d}A \\
[C^2] &\equiv [Y]^{-T}[G]^T \int_A [N]^T[[P(0,1,0)][0]][\tilde{Q}][S]\,\mathrm{d}A \\
[C^3] &\equiv [Y]^{-T}[G]^T \int_A [N]^T[[P(0,1,0)][0]][y\{\tilde{Q}_1\}\ z\{\tilde{Q}_1\}]\,\mathrm{d}A \\
[C^4] &\equiv \int_A [S]^T[y\{\tilde{Q}_3\}\ z\{\tilde{Q}_3\}]\,\mathrm{d}A \\
[C^5] &\equiv \int_A \tilde{Q}_{31} \begin{bmatrix} y^2 & yz \\ yz & z^2 \end{bmatrix} \mathrm{d}A \\
[C^6] &\equiv [Y]^{-T}[G]^T \int_A [N]^T[[P(0,0,1)][0]][\tilde{Q}][S]\,\mathrm{d}A \\
[C^7] &\equiv [Y]^{-T}[G]^T \int_A [N]^T[[P(0,0,1)][0]][y\{\tilde{Q}_1\}\ z\{\tilde{Q}_1\}]\,\mathrm{d}A \\
[C^8] &\equiv \int_A [S]^T[y\{\tilde{Q}_2\}\ z\{\tilde{Q}_2\}]\,\mathrm{d}A \\
[C^9] &\equiv \int_A \tilde{Q}_{21} \begin{bmatrix} y^2 & yz \\ yz & z^2 \end{bmatrix} \mathrm{d}A,
\end{aligned} \tag{51c}
$$

where $[\Phi]$, $[C^0]$, $[C^2]$ and $[C^6]$ are 6×6 matrices, $[E]$, $[C^1]$, $[C^3]$, $[C^4]$, $[C^7]$ and $[C^8]$ are 6×2 matrices, $[F]$, $[C^5]$ and $[C^9]$ are 2×2 matrices, $[A]$, $[B]$ and $[D]$ are 3×3 matrices, $[E^1]$ and $[E^2]$ are 3×2 matrices, and $[\Phi]$, $[F]$, $[C^5]$, $[C^9]$, $[A]$ and $[D]$ are symmetric matrices. The $\bar{F}_i$ and $\bar{M}_i$ are due to nontrivial values of γ'_5 and γ'_6; M_2 and M_3 are the bending moments due to σ_{11}; m_1, m_2 and m_3 are nonlinear twisting and bending moments due to warpings. We note from the expression of $[S]$ in eqn (28b) and eqns (51c), (28a) and (26b) that the influences of warpings, initial curvatures, and shear rotations on the structural stiffnesses are fully accounted for. Although $[\hat{K}]$, $[\bar{X}]$ and $[S]$ are functions of the curvatures ρ_i, one can use $k_i = \rho_i$ in $[S]$ to simplify the formulation without much loss of accuracy. We note that the introduction of these forces, moments and higher-order quantities is essential for the reduction of the three-dimensional problem to a one-dimensional one.

Moreover, it follows from eqns (19), (18) and (33) that

$$\begin{aligned}\int_0^L H\delta\rho_1\,ds &= \int_0^L (-H'\delta\theta_1 - H\rho_3\delta\theta_2 + H\rho_2\delta\theta_3)\,ds + H\delta\theta_1|_0^L\\ \int_0^L H\delta\rho_2\,ds &= \int_0^L (H\rho_3\delta\theta_1 - H'\delta\theta_2 - H\rho_1\delta\theta_3)\,ds + H\delta\theta_2|_0^L\\ \int_0^L H\delta\rho_3\,ds &= \int_0^L (-H\rho_2\delta\theta_1 + H\rho_1\delta\theta_2 - H'\delta\theta_3)\,ds + H\delta\theta_3|_0^L,\end{aligned} \tag{52a}$$

where H denotes a stress moment.

Substituting eqns (26b) and (52) into eqn (50) and using the fact that variations of the initial curvatures are zero, we obtain variation of the elastic energy in terms of the stress-resultants, stress-couples, curvatures and axial strain as

$$\begin{aligned}\delta V = \int_0^L \{&\tilde{F}_1\delta e - (M'_2 - \tilde{F}_3)\delta\gamma_5 + (M'_3 + \tilde{F}_2)\delta\gamma_6 - (\tilde{M}'_1 + \tilde{M}_3\rho_2 - \tilde{M}_2\rho_3)\delta\theta_1\\ &- (\tilde{M}'_2 - \tilde{M}_3\rho_1 + \tilde{M}_1\rho_3)\delta\theta_2 - (\tilde{M}'_3 + \tilde{M}_2\rho_1 - \tilde{M}_1\rho_2)\delta\theta_3\}\,ds\\ &+ [\tilde{M}_1\delta\theta_1 + \tilde{M}_2\delta\theta_2 + \tilde{M}_3\delta\theta_3 + M_2\delta\gamma_5 - M_3\delta\gamma_6]_0^L,\end{aligned} \tag{53}$$

where

$$\begin{aligned}&\tilde{M}_1 \equiv \hat{M}_1 + \bar{M}_1 + m_1, \quad \tilde{M}_2 \equiv \hat{M}_2 + \bar{M}_2 + m_2, \quad \tilde{M}_3 \equiv \hat{M}_3 + \bar{M}_3 + m_3\\ &\tilde{F}_1 \equiv \hat{F}_1 + \bar{F}_1, \quad \tilde{F}_2 \equiv \hat{F}_2 + \bar{F}_2, \quad \tilde{F}_3 \equiv \hat{F}_3 + \bar{F}_3.\end{aligned} \tag{54}$$

It follows from eqns (11b) and (16) that

$$1+e = T_{11}(1+u' - vk_3 + wk_2) + T_{12}(v' + uk_3 - wk_1) + T_{13}(w' - uk_2 + vk_1). \tag{55}$$

Substituting eqn (11b) into (55) and taking the variation, we obtain

$$\delta e = T_{11}\delta(1+u' - vk_3 + wk_2) + T_{12}\delta(v' + uk_3 - wk_1) + T_{13}\delta(w' - uk_2 + vk_1). \tag{56}$$

Using eqns (11a) and (11b), we find that

$$\delta\mathbf{i}_1 = \frac{1}{1+e}[(\delta u' - k_3\delta v + k_2\delta w)\mathbf{i}_x + (\delta v' + k_3\delta u - k_1\delta w)\mathbf{i}_y + (\delta w' - k_2\delta u + k_1\delta v)\mathbf{i}_z] - \frac{\delta e}{1+e}\mathbf{i}_1. \tag{57}$$

Then, using eqns (33), (57) and (7a), we obtain

$$
\begin{aligned}
-(1+e)\delta\theta_2 &= (1+e)\mathbf{i}_3\cdot\delta\mathbf{i}_1 \\
&= T_{31}(\delta u'-k_3\delta v+k_2\delta w)+T_{32}(\delta v'+k_3\delta u-k_1\delta w) \\
&\quad +T_{33}(\delta w'-k_2\delta u+k_1\delta v)
\end{aligned} \tag{58a}
$$

$$
\begin{aligned}
(1+e)\delta\theta_3 &= (1+e)\mathbf{i}_2\cdot\delta\mathbf{i}_1 \\
&= T_{21}(\delta u'-k_3\delta v+k_2\delta w)+T_{22}(\delta v'+k_3\delta u-k_1\delta w) \\
&\quad +T_{23}(\delta w'-k_2\delta u+k_1\delta v).
\end{aligned} \tag{58b}
$$

Substituting eqn (56), $\tilde{F}_3\times$ (58a), and $\tilde{F}_2\times$ (58b) into eqn (53) and then integrating by parts, we obtain

$$
\begin{aligned}
\delta V = \int_0^L \Bigg\{ & -\frac{\partial}{\partial s}(\{\tilde{F}_1,\tilde{F}_2,\tilde{F}_3\}[T])\begin{Bmatrix}\delta u\\ \delta v\\ \delta w\end{Bmatrix}+\{\tilde{F}_1,\tilde{F}_2,\tilde{F}_3\}[T][k]^T\begin{Bmatrix}\delta u\\ \delta v\\ \delta w\end{Bmatrix} \\
& -[\tilde{M}_1'+\tilde{M}_3\rho_2-\tilde{M}_2\rho_3]\delta\theta_1-[\tilde{M}_2'-\tilde{M}_3\rho_1+\tilde{M}_1\rho_3 \\
& -(1+e)\tilde{F}_3]\delta\theta_2-[\tilde{M}_3'+\tilde{M}_2\rho_1-\tilde{M}_1\rho_2+(1+e)\tilde{F}_2]\delta\theta_3 \\
& -(M_2'-\tilde{F}_3)\delta\gamma_5+(M_3'+\tilde{F}_2)\delta\gamma_6\Bigg\}\,\mathrm{d}s \\
& +\left[\{\tilde{F}_1,\tilde{F}_2,\tilde{F}_3\}[T]\begin{Bmatrix}\delta u\\ \delta v\\ \delta w\end{Bmatrix}+\tilde{M}_1\delta\theta_1+\tilde{M}_2\delta\theta_2+\tilde{M}_3\delta\theta_3+M_2\delta\gamma_5-M_3\delta\gamma_6\right]_0^L .
\end{aligned} \tag{59}
$$

4.3. *Equations of motion*

Substituting eqns (45) and (59) into (30), using eqn (52a), and setting each of the coefficients of δu, δv, δw, $\delta\theta_1$, $\delta\gamma_5$, $\delta\gamma_6$, $\delta\theta_2$ and $\delta\theta_3$ equal to zero, we obtain the following equations of motion;

$$
\frac{\partial}{\partial s}\left([T]^T\begin{Bmatrix}\tilde{F}_1\\ \tilde{F}_2\\ \tilde{F}_3\end{Bmatrix}\right)-[k][T]^T\begin{Bmatrix}\tilde{F}_1\\ \tilde{F}_2\\ \tilde{F}_3\end{Bmatrix}=\begin{Bmatrix}A_u+\mu_1\dot{u}\\ A_v+\mu_2\dot{v}\\ A_w+\mu_3\dot{w}\end{Bmatrix} \tag{60}
$$

$$\tilde{M}_1'+\tilde{M}_3\rho_2-\tilde{M}_2\rho_3 = A_{\theta_1}-A_{\rho_1}'+\mu_4\dot{\phi} \tag{61}$$

$$M_2'-\tilde{F}_3 = A_{\gamma_5}+\mu_5\dot{\gamma}_5 \tag{62}$$

$$-M_3'-\tilde{F}_2 = A_{\gamma_6}+\mu_6\dot{\gamma}_6 \tag{63}$$

$$\tilde{M}_2'-\tilde{M}_3\rho_1+\tilde{M}_1\rho_3-(1+e)\tilde{F}_3 = A_{\theta_2}-A_{\rho_1}\rho_3 \tag{64}$$

$$\tilde{M}_3'+\tilde{M}_2\rho_1-\tilde{M}_1\rho_2+(1+e)\tilde{F}_2 = A_{\theta_3}+A_{\rho_1}\rho_2. \tag{65}$$

Here, we added a linear viscous damping term to each of eqns (60)–(63), which correspond to δW_{nc} in eqn (30), where μ_1, μ_2, μ_3, μ_4, μ_5 and μ_6 are damping coefficients per unit length corresponding to u, v, w, ϕ, γ_5 and γ_6, respectively. The boundary conditions for the beam are of the form

$$\begin{aligned}
\delta u = 0 &\quad \text{or} \quad \tilde{F}_1 T_{11} + \tilde{F}_2 T_{21} + \tilde{F}_3 T_{31} \quad \text{specified} \\
\delta v = 0 &\quad \text{or} \quad \tilde{F}_1 T_{12} + \tilde{F}_2 T_{22} + \tilde{F}_3 T_{32} \quad \text{specified} \\
\delta w = 0 &\quad \text{or} \quad \tilde{F}_1 T_{13} + \tilde{F}_2 T_{23} + \tilde{F}_3 T_{33} \quad \text{specified} \\
\delta \theta_1 = 0 &\quad \text{or} \quad \tilde{M}_1 + A_{\rho_1} \quad \text{specified} \\
\delta \gamma_5 = 0 &\quad \text{or} \quad M_2 \quad \text{specified} \\
\delta \gamma_6 = 0 &\quad \text{or} \quad M_3 \quad \text{specified} \\
\delta \theta_2 = 0 &\quad \text{or} \quad \tilde{M}_2 \quad \text{specified} \\
\delta \theta_3 = 0 &\quad \text{or} \quad \tilde{M}_3 \quad \text{specified.}
\end{aligned} \tag{66}$$

Because $\delta\theta_2$ and $\delta\theta_3$ are functions of δu, δv, δw and their first derivatives with respect to s [see eqns (58a) and (58b)] and no extra dependent variables describing the rotations with respect to the η and ζ axes are involved, eqns (64) and (65) are statements of the balance of moments with respect to the η and ζ axes. Hence, they can be used to solve for the transverse shear forces $\tilde{F}_2$ and $\tilde{F}_3$. The result is

$$\tilde{F}_3 = \frac{1}{1+e}(\tilde{M}'_2 - \tilde{M}_3\rho_1 + \tilde{M}_1\rho_3 - A_{\theta_2} + A_{\rho_1}\rho_3) \tag{67}$$

$$\tilde{F}_2 = \frac{-1}{1+e}(\tilde{M}'_3 + \tilde{M}_2\rho_1 - \tilde{M}_1\rho_2 - A_{\theta_3} - A_{\rho_1}\rho_2). \tag{68}$$

Substituting eqns (67) and (68) into (62) and (63), using (18b), and then putting eqns (60)–(63) into matrix form, we obtain

$$[T]^T\left(\begin{Bmatrix}\tilde{F}'_1\\ \tilde{F}'_2\\ \tilde{F}'_3\end{Bmatrix} - [\mathrm{K}]\begin{Bmatrix}\tilde{F}_1\\ \tilde{F}_2\\ \tilde{F}_3\end{Bmatrix}\right) = \begin{Bmatrix}A_u + \mu_1\dot{u}\\ A_v + \mu_2\dot{v}\\ A_w + \mu_3\dot{w}\end{Bmatrix} \tag{69a}$$

$$\begin{Bmatrix}\tilde{M}'_1\\ \tilde{M}'_2\\ \tilde{M}'_3\end{Bmatrix} - [\mathrm{K}]\begin{Bmatrix}\tilde{M}_1\\ \tilde{M}_2\\ \tilde{M}_3\end{Bmatrix} - (1+e)\begin{Bmatrix}0\\ M'_2\\ M'_3\end{Bmatrix} = \begin{Bmatrix}A_{\theta_1} - A'_{\rho_1} + \mu_4\dot{\phi}\\ A_{\theta_2} - A_{\rho_1}\rho_3 - (1+e)(A_{\gamma_5} + \mu_5\dot{\gamma}_5)\\ A_{\theta_3} + A_{\rho_1}\rho_2 + (1+e)(A_{\gamma_6} + \mu_6\dot{\gamma}_6)\end{Bmatrix}. \tag{69b}$$

We note that eqn (69b) can also be obtained from eqns (61), (64) and (65) by using eqns (62) and (63) to eliminate $\tilde{F}_2$ and $\tilde{F}_3$ from eqns (64) and (65).

The expanded form of $\delta\theta_1$ is also needed in the boundary conditions, which is obtained from eqns (33), (7a, b), (17a), (57) and (11b), the orthonormality of the $\mathbf{i}_j$, and $\mathbf{i}_1 \cdot \delta\mathbf{i}_1 = 0$ as

$$\begin{aligned}
\delta\theta_1 &= \mathbf{i}_3 \cdot \delta\mathbf{i}_2 = \delta\phi + \frac{1}{1+T_{11}}(T_{13}\delta T_{12} - T_{12}\delta T_{13}) \\
&= \delta\phi + \frac{1}{(1+e)(1+T_{11})}[(T_{13}k_3 + T_{12}k_2)\delta u - T_{12}k_1\delta v - T_{13}k_1\delta w + T_{13}\delta v' - T_{12}\delta w'].
\end{aligned} \tag{70}$$

We note that the equations of motion [i.e. eqns (69a) and (69b)] are nonlinearly coupled through the components of the transformation matrix $[T]$ and the curvatures ρ_i, which are due to geometric nonlinearities. Also, it can be seen from eqns (69), (51a), (51b), (29) and (19) that the equations of motion are linearly coupled due to anisotropy and initial curvatures. Moreover, the boundary conditions are also coupled due to geometric nonlinearities, anisotropy and initial curvatures, as shown in eqns (66), (58) and (70).

Substituting eqns (12), (11b) and (17a) into eqn (7b), one can obtain exact expressions for the elements of the matrix $[T]$ in terms of u, v, w, ϕ and their derivatives. Substituting these expressions into eqn (19), one can obtain exact expressions for the curvatures in terms of u, v, w, ϕ and their derivatives. Moreover, the inertial terms in eqn (45) are exact, except that the inertias due to in-plane warpings are neglected.

5. DISCUSSIONS AND APPLICATIONS

5.1. *Characteristics of the model*

Because the equations of motion (60)–(65) consist of eight first-order equations with respect to s subject to eight boundary conditions, it is called an eighth-order system. However, substituting for the $\delta\theta_i$ from eqns (70), (58a) and (58b) into eqns (45) and (59), then substituting the results into eqn (30), and setting each of the coefficients of δu, δv, δw, $\delta\phi$, $\delta u'$, $\delta v'$, $\delta w'$, $\delta\gamma_5$ and $\delta\gamma_6$ equal to zero, one obtains nine equations subject to nine boundary conditions, implying that the system is of ninth order, which is not true. Consequently, we conclude that $\delta u'$, $\delta v'$, $\delta w'$ and $\delta\phi$ are not independent, and that they are related in such a way that there are only three independent rotational variations, which are $\delta\theta_1$, $\delta\theta_2$ and $\delta\theta_3$, as shown in eqns (70), (58a) and (58b).

There are no integral terms in the present formulation and the equations are symmetric and interchangeable because only two Euler angles were used. If three Euler angles were used in obtaining the transformation matrix $[T]$, then the resulting equations of motion either are asymmetric or contain integral terms (Pai and Nayfeh, 1990).

Certain small offsets of the blade axis are often provided to reduce steady blade-bending stresses, to improve rotorcraft flying qualities, or to enhance rotor blade aeroelastic stability (Hodges, 1976). All the effects of initial bending and twisting curvatures, precone, droop, sweep, torque offset and blade root offset can be included in the model by properly choosing the functions $A(s)$, $B(s)$, $C(s)$, $\theta_{21}(s)$, $\theta_{22}(s)$ and $\theta_{23}(s)$.

Blade root feathering flexibility, which is due to the flexibility of the pitch-link and the control system and hence permit rigid-body rotation of the blade with respect to the pitch-bearing axis, can be included by using the boundary condition

$$\tilde{M}_1 + A_{\rho_1} = \mathrm{K}_\phi \phi \tag{71}$$

at the blade root, where K_ϕ denotes the equivalent torsional spring constant for the pitch link.

Using eqns (6a) and (18a), we put eqns (60), (61), (64) and (65) in vector form as

$$\frac{\partial \mathbf{F}}{\partial s} = (A_u + \mu_1 \ddot{u})\mathbf{i}_x + (A_v + \mu_2 \ddot{v})\mathbf{i}_y + (A_w + \mu_3 \ddot{w})\mathbf{i}_z \tag{72}$$

$$\frac{\partial \mathbf{M}}{\partial s} + (1+e)\mathbf{i}_1 \times \mathbf{F} = (A_{\theta_1} - A'_{\rho_1} + \mu_4 \ddot{\phi})\mathbf{i}_1 + (A_{\theta_2} - A_{\rho_1}\rho_3)\mathbf{i}_2 + (A_{\theta_3} + A_{\rho_1}\rho_2)\mathbf{i}_3, \tag{73}$$

where

$$\mathbf{F} = \tilde{F}_1\mathbf{i}_1 + \tilde{F}_2\mathbf{i}_2 + \tilde{F}_3\mathbf{i}_3, \quad \mathbf{M} = \tilde{M}_1\mathbf{i}_1 + \tilde{M}_2\mathbf{i}_2 + \tilde{M}_3\mathbf{i}_3. \tag{74}$$

Equations (72) and (73) have the same form as those directly obtained from the Newtonian approach (Pai and Nayfeh, 1990) except that the F_i and M_i are replaced by the $\tilde{F}_i$ and $\tilde{M}_i$, respectively. However, the differences between the $\tilde{F}_i$ and F_i and $\tilde{M}_i$ and M_i, which are due to in-plane and out-of-plane warpings, cannot be easily determined and included in the

Newtonian approach. Moreover, eqns (62) and (63) cannot easily be obtained directly from the Newtonian approach.

5.2. *Simplified beam theories*

For most beam theories, in-plane warpings W_2 and W_3 are neglected. In this case, it follows from eqn (25) that

$$W_2 = W_3 = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{23} = 0. \tag{75}$$

But, for beams with no transverse distributed loads, the assumptions

$$\sigma_{22} = \sigma_{33} = \sigma_{23} = 0 \tag{76}$$

are more reasonable than those in eqn (75) (Pai and Nayfeh, 1992). However, both theories based on the assumptions (75) and (76) do not fully account for three-dimensional stress effects because $\sigma_{22}\delta\varepsilon_{22}+\sigma_{23}\delta\varepsilon_{23}+\sigma_{33}\delta\varepsilon_{33} = 0$ in eqn (31b), and hence they are only valid for very slender isotropic beams.

If the small nonlinear moments m_i [see eqn (51b)] due to warpings are neglected in eqn (54), we obtain from eqns (54), (51a) and (51c) that

$$\begin{aligned} &\{\tilde{F}_1, \tilde{F}_2, \tilde{F}_3\}^T = [\tilde{A}]\{\tilde{\psi}\}, \quad \{\tilde{M}_1, \tilde{M}_2, \tilde{M}_3\}^T = [\tilde{D}]\{\tilde{\psi}\} \\ &[\tilde{A}] \equiv [[A][B][E^1]], \quad [\tilde{D}] = [[B]^T[D][E^2]] \\ &\{\tilde{\psi}\} \equiv \{e, \gamma_6, \gamma_5, \rho_1 - k_1, \rho_2 - k_2, \rho_3 - k_3, \gamma_6', \gamma_5'\}^T, \end{aligned} \tag{77a}$$

where the structural stiffness matrices $[\tilde{A}]$ and $[\tilde{D}]$ are 3×8 matrices. Then, by using eqns (77a) and (51a), we rewrite the governing equations (69a, b) in terms of $\{\tilde{\psi}\}$ as

$$[T]^T\{([\tilde{A}]\{\tilde{\psi}\})' - [\mathrm{K}][\tilde{A}]\{\tilde{\psi}\}\} = \begin{Bmatrix} A_u + \mu_1 \ddot{u} \\ A_v + \mu_2 \ddot{v} \\ A_w + \mu_3 \ddot{w} \end{Bmatrix} \tag{77b}$$

$$([\hat{D}]\{\tilde{\psi}\})' - [\mathrm{K}][\hat{D}]\{\tilde{\psi}\} = \begin{Bmatrix} A_{\theta_1} - A'_{\rho_1} + \mu_4 \ddot{\phi} \\ A_{\theta_2} - A_{\rho_1}\rho_3 - (1+e)(A_{\gamma_5} + \mu_5 \ddot{\gamma}_5) \\ A_{\theta_3} + A_{\rho_1}\rho_2 + (1+e)(A_{\gamma_6} + \mu_6 \ddot{\gamma}_6) \end{Bmatrix}, \tag{77c}$$

where

$$[\hat{D}] \equiv [\tilde{D}] - (1+e)\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}[[E]^T[F]]. \tag{77d}$$

We point out here that $[\tilde{A}]$ and $[\tilde{D}]$ are functions of the initial curvatures k_i and hence $[\tilde{A}]'$ and $[\hat{D}]'$ are nontrivial, and the fully nonlinear expressions of e and ρ_i are shown in eqns (12) and (19). We note that, because the $[S]$ in eqn (28b) includes the effects due to warpings and initial curvatures, the effects of initial curvatures, warpings, and three-dimensional stresses are included in the calculation of stiffness matrices $[A]$, $[B]$, $[D]$ and $[E]$. The elastic terms in the governing eqns (77b, c) are functions of u, v, w, ϕ, γ_5 and γ_6, as shown in Appendix B.

If m_i, γ_5' and γ_6' are neglected in eqn (51a), it follows from eqn (51a) and (54) that $\tilde{F}_i = \hat{F}_i$ and $\tilde{M}_i = \hat{M}_i$ and the governing equations (69a, b) reduce to

$$[T]^T\{([[A][B]]\{\psi\})'-[K][[A][B]]\{\psi\}\} = \begin{Bmatrix} A_u+\mu_1\ddot{u} \\ A_v+\mu_2\ddot{v} \\ A_w+\mu_3\ddot{w} \end{Bmatrix} \tag{78a}$$

$$([[B]^T[D]]\{\psi\})'-[K][[B]^T[D]]\{\psi\}-\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}((1+e)[E]^T\{\psi\})'$$

$$= \begin{Bmatrix} A_{\theta_1}-A'_{\rho_1}+\mu_4\ddot{\phi} \\ A_{\theta_2}-A_{\rho_1}\rho_3-(1+e)(A_{\gamma_5}+\mu_5\ddot{\gamma}_5) \\ A_{\theta_3}+A_{\rho_1}\rho_2+(1+e)(A_{\gamma_6}+\mu_6\ddot{\gamma}_6) \end{Bmatrix}. \tag{78b}$$

The total out-of-plane warping $\bar{W}_1$ is given by [see eqn (32b)]

$$\bar{W}_1 = z\gamma_5+y\gamma_6+\{1,0,0\}[N][G][Y]^{-1}\{e,\gamma_6,\gamma_5,\rho_1-k_1,\rho_2-k_2,\rho_3-k_3\}^T.$$

Out-of-plane warpings due to extension and bendings (i.e. e, ρ_2-k_2 and ρ_3-k_3) are very often neglected in beam modeling and hence $\bar{W}_1$ is usually represented by

$$\bar{W}_1 = g_1(\rho_1-k_1)+g_2\gamma_5+g_3\gamma_6,$$

where the g_i are defined in (32d). In most beam theories, the out-of-plane warping due to torsion (i.e. $g_1(\rho_1-k_1)$) is either neglected or approximated by using St Venant's torsional warping solutions for isotropic bars. Treating the out-of-plane shear warpings $g_2\gamma_5+g_3\gamma_6$ in different ways results in different beam theories. In the first-order shear theory, which is called Timoshenko's beam theory in the case of planar vibrations, one assumes that $g_2 = z$ and $g_3 = y$ (i.e. $W_1 = 0$). In the third-order shear theory, one assumes that $g_2 = z-4z^2/3h^2$ and $g_3 = y-4y^3/3\hat{b}^2$ (i.e. $W_1 = -\gamma_5 4z^3/3h^2-\gamma_6 4y^3/3\hat{b}^2$) in the case of rectangular cross-sections with width $\hat{b}$ and thickness h.

In the Euler–Bernoulli beam theory, it is assumed that

$$W_i = g_i = \gamma_5 = \gamma_6 = 0, \quad i = 1,2,3 \tag{79}$$

and hence all warpings are neglected. It follows from eqns (25), (26a, b) and (28b) that

$$\begin{aligned} &\varepsilon_{11} = e+z(\rho_2-k_2)-y(\rho_3-k_3) \\ &\varepsilon_{12} = -z(\rho_1-k_1), \quad \varepsilon_{13} = y(\rho_1-k_1), \quad \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{23} = 0 \end{aligned} \tag{80a}$$

and

$$\{\varepsilon\} = [S]\{\psi\}, \quad [S] = [X], \tag{80b}$$

where the expression of $[X]$ is shown in eqns (16b) and (1f). Hence, it follows from eqns (79), (80b), (51a–c), (54) and (29) that

$$\tilde{F}_i = F_i, \quad \tilde{M}_i = M_i, \quad i = 1,2,3, \tag{81}$$

where F_i and M_i are defined in eqn (27c).

For flexible beams, because the inertias are mainly due to the global displacements u, v, w and ϕ and inertias due to warpings are negligible, the Euler–Bernoulli assumptions [see eqn (79)] are appropriate for obtaining inertia terms. Substituting eqn (79) into (48) and (49) yields

$$f_{0i} = f_{1i} = f_{2i} = f_{3i} = f_{4i} = f_{5i} = 0 \quad \text{for } i = 1,2,3$$
$$I_{jk} = 0 \quad \text{for } j > 3 \text{ and/or } k > 3 \tag{82a}$$

and hence

$$[I_1] = \begin{bmatrix} 0 & I_{31} & -I_{21} \\ -I_{31} & 0 & 0 \\ I_{21} & 0 & 0 \end{bmatrix}, \quad [I_2] = \begin{bmatrix} I_{22}+I_{33} & 0 & 0 \\ 0 & I_{33} & -I_{23} \\ 0 & -I_{23} & I_{22} \end{bmatrix}, \quad [I_3] = [0] \tag{82b}$$

$$A_{\rho_1} = A_{\gamma_5} = A_{\gamma_6} = 0. \tag{82c}$$

Thus, the rotary inertia terms in the governing eqns (69a, b) are drastically simplified. Moreover, if the axis x passes through the mass centroid and the y and z axes are the principal axes of the mass inertias, then

$$I_{21} = I_{31} = I_{23} = 0, \quad [I_1] = [0]. \tag{83}$$

For hover conditions

$$U = V = W = 0 \tag{84a}$$

and the rotation axis is always along the c axis (see Fig. 2). Hence, $\mathbf{i}_h = \mathbf{i}_c = \mathbf{i}_{\bar{c}}$, and the transformation matrix $[T^a]$ in eqn (38) reduces to

$$[T^a] = \begin{bmatrix} \cos\Theta_3 & \sin\Theta_3 & 0 \\ -\sin\Theta_3 & \cos\Theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Theta_3 = \int_0^t \Omega(t)\,\mathrm{d}t. \tag{84b}$$

Moreover, it follows from eqns (36b–d) and (5a, b) that

$$\omega_1^a = \omega_2^a = 0, \quad \omega_3^a = \Omega$$
$$\omega_1^x = C'\Omega, \quad \omega_2^x = \cos\theta_{23}\Omega, \quad \omega_3^x = (A'\cos\theta_{22} - B'\cos\theta_{21})\Omega \tag{84c}$$

and ω_i are shown in eqn (43). The equations of motion and boundary conditions for initially curved and twisted Euler–Bernoulli beams under hover conditions are shown in Appendix C.

If, furthermore, we assume that the reference frame abc is fixed (i.e. a cantilever beam), then

$$[T^a] = [I], \quad U = V = W = \Omega = \dot{\omega}^a = \dot{\omega}^x = 0. \tag{85a}$$

Hence, it follows from eqns (46), (82a–c), (83) and (85a) that $\{Q_1\} = \{0\}$ and $\{Q_2\} = \{\ddot{u}, \ddot{v}, \ddot{w}\}^T$ and hence

$$\begin{Bmatrix} A_u \\ A_v \\ A_w \end{Bmatrix} = I_{11}\{Q_2\} = I_{11}\begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{Bmatrix} \tag{85b}$$

$$\begin{Bmatrix} A_{\theta_1} \\ A_{\theta_2} \\ A_{\theta_3} \end{Bmatrix} = [I_2]\{\dot{\omega}\} - [P(\omega)][I_2]\{\omega\} = \begin{Bmatrix} \bar{I}_{22}\dot{\omega}_1 - (I_{33} - I_{22})\omega_2\omega_3 \\ I_{33}\dot{\omega}_2 - (I_{22} - \bar{I}_{22})\omega_1\omega_3 \\ I_{22}\dot{\omega}_3 - (\bar{I}_{22} - I_{33})\omega_1\omega_2 \end{Bmatrix}, \tag{85c}$$

where $\bar{I}_{22} \equiv I_{22}+I_{33}$. Equations (85c) have the same form as the Euler equations used in rigid-body dynamics.

5.3. *Comparisons*

The engineering stresses σ_{ij} and engineering strains ε_{ij} in eqn (29) are geometric measures and are defined with respect to the deformed coordinate system, which means that it is appropriate to use a constant material stiffness matrix $[\tilde{Q}]$. On the other hand, if the second Piola–Kirchhoff stresses and Green–Lagrange strains are used, deformation-dependent material stiffness matrices need to be used to relate such energy-related stress and strain measures.

Furthermore, because of the use of local stress and strain measures, the structural stiffness matrices, which relate the local internal stress resultants $\tilde{F}_i$ and moments $\tilde{M}_i$ to the force strains e, γ_5, γ_6 and moment strains ρ_i (i.e. curvatures), are obtained by direct integration [see eqns (51a, c)]. On the other hand, if "global" stress and strain measures (i.e. stresses and strains defined with respect to the undeformed coordinate system) were used, one would need to express the total strain energy U of the beam in terms of the force and moment strains by using the "global" stresses and strains, define the local internal stress resultants and moments as the derivatives of U with respect to the force and moment strains, and then take the derivatives of U to obtain the structural stiffness matrices. This method was used by Hodges (1990), Atilgan and Hodges (1991) and Simo and Vu-Quoc (1991). We point out here that the stress resultants and moments obtained by using this method cannot correctly account for the influence of warpings [see eqns (54) and (51b)] and do not include the effects of γ_5' and γ_6' [see eqns (54) and (51a)].

In this paper, the virtual rotations are defined by using the unit vectors of the deformed local coordinate system $\xi\eta\zeta$ and can be represented exactly in terms of the variations of the displacements and their derivatives, as shown in eqns (70) and (58a, b). Moreover, in obtaining the relations between the variations of the curvatures and the virtual rotations, we did not use the Kirchhoff kinetic analogy, which was used by most of researchers.

The equations of motion (69a, b) are similar to some other beam equations, especially those of Hodges (1990), Atilgan and Hodges (1991) and Simo and Vu-Quoc (1991), but not the same. The stress resultant $\tilde{F}_1$ in our theory is tangent to the deformed reference line, but, in the theories of Hodges (1990), Atilgan and Hodges (1991) and Simo and Vu-Quoc (1991), F_1 is not tangent to the deformed reference line because the location of the local coordinate system is influenced by the shear rotations. The Jaumann–Biot–Cauchy strains used by Hodges (1990) and Atilgan and Hodges (1991) are very similar to our local engineering strains in eqns (26a) and (26b), but their strains are defined and explained [see eqn (18) of Danielson and Hodges (1987); eqn (14) of Hodges (1990); eqn (26) of Atilgan and Hodges (1991)] to be with respect to the undeformed coordinate system. That is why they need to use the strains energy U to define the local internal stress resultants and moments as the derivatives of U with respect to the force and moment strains.

5.4. *Applications*

The present curved and twisted beam theory can be applied to helical springs. It follows from Fig. 6 that the unloaded position vector $\mathbf{R}$ of the reference point of the observed cross-section is given by

$$\mathbf{R} = r\cos\theta\mathbf{i}_a + r\sin\theta\mathbf{i}_b + r\theta\tan\psi\mathbf{i}_c, \tag{86}$$

where r is the radius of the projection of the reference line onto the a–b plane and ψ is the pitch angle; both r and ψ are assumed to be constant. Hence, we have

$$\mathbf{i}_x = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s} = -r\theta'\sin\theta\mathbf{i}_a + r\theta'\cos\theta\mathbf{i}_b + r\theta'\tan\psi\mathbf{i}_c \tag{87a}$$

$$\mathbf{i}_y = \cos\theta\mathbf{i}_a + \sin\theta\mathbf{i}_b \tag{87b}$$

and

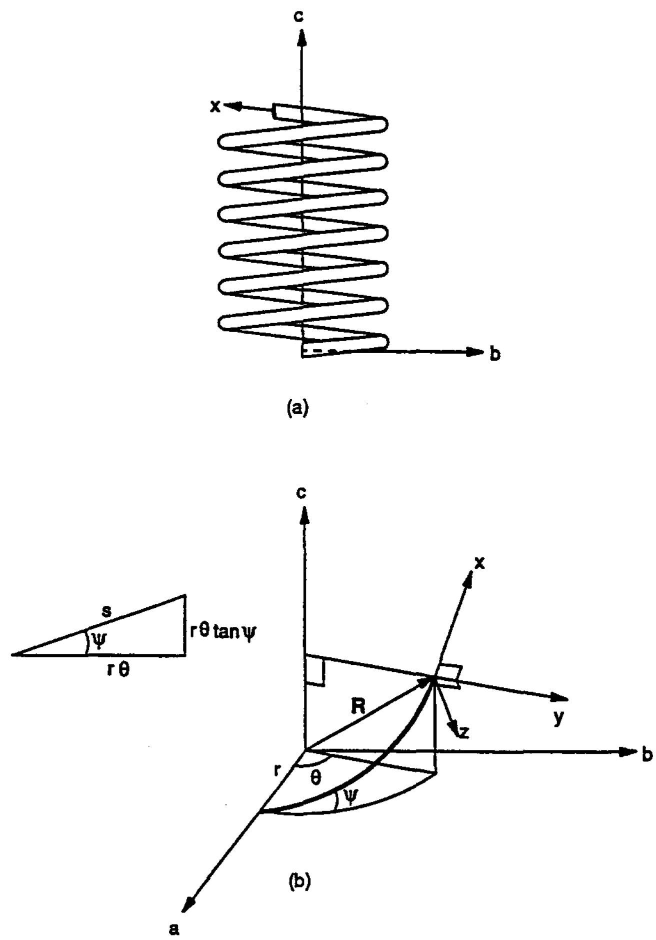

Fig. 6(a). A helical spring, and (b) two coordinate systems *abc* and *xyz* and their relationship.

$$[T^x] = \begin{bmatrix} -r\theta' \sin\theta & r\theta' \cos\theta & r\theta' \tan\psi \\ \cos\theta & \sin\theta & 0 \\ -r\,\theta' \sin\theta \tan\psi & r\theta' \cos\theta \tan\psi & -r\theta' \end{bmatrix}. \tag{88}$$

Using eqns (88) and (6c) and the identity $s \cos\psi = r\theta$, we find that the initial curvatures are

$$k_1 = \frac{1}{r}\cos\psi \sin\psi, \quad k_2 = 0, \quad k_3 = -\frac{1}{r}\cos^2\psi. \tag{89}$$

Substituting eqns (88) and (89) into eqns (C1a, b), one can obtain the Euler–Bernoulli beam model of helical springs.

For circular rings, $\psi = 0$ and the initial curvatures are obtained from eqn (89) as

$$k_1 = k_2 = 0, \quad k_3 = -\frac{1}{r}. \tag{90}$$

For straight beams, the initial curvatures k_i are zero.

6. CLOSURE

A new methodology that combines the dynamics of particles, exact coordinate transformations, the new concept of local engineering stress and strain measures and virtual local rotations, and the extended Hamilton principle is used to develop a geometrically exact nonlinear curved beam model for solid composite rotor blades undergoing large vibrations in three-dimensional space. The six nonlinear equations of motion describing one extension, two bending, one torsion, and two shearing vibrations are linearly coupled due to structural anisotropy and initial curvatures and are nonlinearly coupled due to large rotations. The influence of in-plane and out-of-plane warpings and three-dimensional stress effects on the elastic properties are accounted for by using central solutions obtained from a two-dimensional, static, sectional, finite element analysis. Also, extensionality and the initial curvatures are fully accounted for and the theory contains most of beam theories as special cases. The derivation enables one to gain additional insight into the physical meaning of all structural and inertial terms and the relation between the energy and Newtonian formulations. The fully nonlinear equations of motion are expressed in matrix form as eqns (69a) and (69b).

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APPENDIX A: THE CONCEPT OF LOCAL ENGINEERING STRAINS

Here we show how local engineering strains can be obtained by using the concept of local displacements. Figure A1 shows the undeformed configuration of the filament $\overline{AB}$ and its deformed configuration $\overline{A'B'}$ in the absence of in-plane and out-of-plane warpings. The position vectors of points A and B are given by

$$\mathbf{R}_A = \mathbf{R}_0 + y\mathbf{i}_y + z\mathbf{i}_z$$
$$\mathbf{R}_B = \mathbf{R}_A + \frac{\partial \mathbf{R}_A}{\partial s}\,\mathrm{d}s = \mathbf{R}_A + [(1 - yk_3 + zk_2)\mathbf{i}_x - zk_1\mathbf{i}_y + yk_1\mathbf{i}_z]\,\mathrm{d}s, \tag{A1}$$

where $\partial\mathbf{R}_0/\partial s = \mathbf{i}_x$ and eqns (6a, b) are used. Hence, the length $\mathrm{d}\hat{s}$ of $\overline{AB}$ and the unit vector $\mathbf{i}_{\hat{x}}$ along the $\hat{x}$ axis are

$$\mathrm{d}\hat{s} = |\mathbf{R}_B - \mathbf{R}_A| = \tau\,\mathrm{d}s, \quad \mathbf{i}_{\hat{x}} = \frac{\overrightarrow{AB}}{\mathrm{d}\hat{s}} = \frac{1}{\tau}[(1 - yk_3 + zk_2)\mathbf{i}_x - zk_1\mathbf{i}_y + yk_1\mathbf{i}_z], \tag{A2}$$

where

$$\tau \equiv \sqrt{(1 - yk_3 + zk_2)^2 + (zk_1)^2 + (yk_1)^2}.$$

We note that $\mathbf{i}_x = \mathbf{i}_{\hat{x}}$ only if the initial twisting curvature $k_1 = 0$.

The displacement vector $\mathbf{D}$ ($= \overrightarrow{AA'}$) of the point A is given by

$$\mathbf{D} = \{u, v, w\}\{\mathbf{i}_{xyz}\} + y\mathbf{i}_2 + z\mathbf{i}_3 - y\mathbf{i}_y - z\mathbf{i}_z. \tag{A3}$$

Taking the derivative of eqn (A3) and using eqns (6a, b), (18a, b) and (A2), we obtain

$$\frac{\partial \mathbf{D}}{\partial s} = \{u', v', w'\}\{\mathbf{i}_{xyz}\} + \{u, v, w\}[k]\{\mathbf{i}_{xyz}\} + y(\rho_1\mathbf{i}_3 - \rho_3\mathbf{i}_1) + z(\rho_2\mathbf{i}_1 - \rho_1\mathbf{i}_2) + \mathbf{i}_x - \tau\mathbf{i}_{\hat{x}}. \tag{A4}$$

It follows from Fig. A1 that $\overrightarrow{BB'} = \partial\mathbf{D}/\partial s\,\mathrm{d}s$ and hence the local axial strain along the ξ axis is

$$\varepsilon_{11} = \frac{\frac{\partial \mathbf{D}}{\partial s}\mathrm{d}s + \mathrm{d}\hat{s}\mathbf{i}_{\hat{x}} - \mathrm{d}\hat{s}\mathbf{i}_{\hat{1}}}{\mathrm{d}\hat{s}}\cdot\mathbf{i}_{\hat{1}} = \frac{1}{\tau}\frac{\partial \mathbf{D}}{\partial s}\cdot\mathbf{i}_{\hat{1}} + \mathbf{i}_{\hat{x}}\cdot\mathbf{i}_{\hat{1}} - 1. \tag{A5}$$

In the absence of in-plane and out-of-plane warpings, the position vectors of points A' and B' are $\mathbf{R}_{A'}$ and $\mathbf{R}_{B'}$, which are given by

$$\mathbf{R}_{A'} = \mathbf{R}_{0'} + y\mathbf{i}_2 + z\mathbf{i}_3$$
$$\mathbf{R}_{B'} = \mathbf{R}_{A'} + \frac{\partial \mathbf{R}_{A'}}{\partial s}\,\mathrm{d}s = \mathbf{R}_{A'} + [(1 + e - y\rho_3 + z\rho_2)\mathbf{i}_1 - z\rho_1\mathbf{i}_2 + y\rho_1\mathbf{i}_3]\,\mathrm{d}s, \tag{A6}$$

where $\partial\mathbf{R}_{0'}/\partial s = (1+e)\mathbf{i}_1$ and eqns (18a, b) are used. Hence, the unit vector $\mathbf{i}_{\hat{1}}$ along the ξ axis is

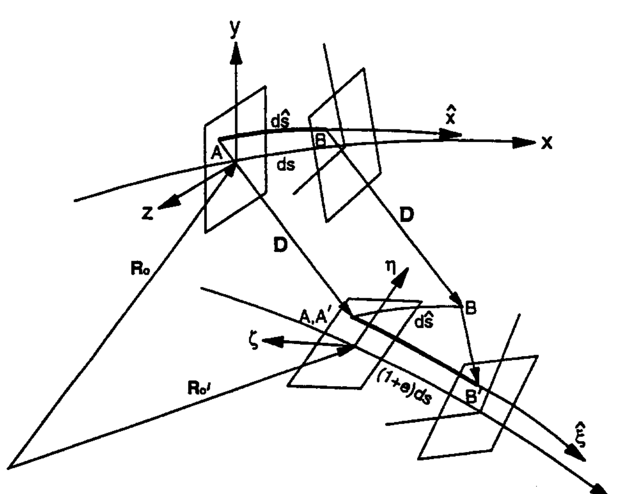

Fig. A1. The geometric relation between the undeformed and deformed configurations of a filament $\overline{AB}$.

$$\mathbf{i}_{\hat{1}} = \frac{1}{\hat{\tau}}[(1+e-y\rho_3+z\rho_2)\mathbf{i}_1 - z\rho_1\mathbf{i}_2 + y\rho_1\mathbf{i}_3], \tag{A7a}$$

where

$$\hat{\tau} \equiv \sqrt{(1+e-y\rho_3+z\rho_2)^2+(z\rho_1)^2+(y\rho_1)^2}.$$

We note that, if the influence of local rotations due to strainable local displacements was neglected, (A7a) would reduce to

$$\mathbf{i}_{\hat{1}} = \frac{1}{\tau}[(1-yk_3+zk_2)\mathbf{i}_1 - zk_1\mathbf{i}_2 + yk_1\mathbf{i}_3] \tag{A7b}$$

which would be a reasonable approximation under the assumption of small strains. Substituting eqns (A7a) and (A4) into eqn (A5) and using eqns (11a, b) and (7a), we obtain

$$\begin{aligned}
\varepsilon_{11} &= \frac{1+e-y\rho_3+z\rho_2}{\tau\hat{\tau}}[(u'-vk_3+wk_2)T_{11}+(v'+uk_3-wk_1)T_{12}+(w'-uk_2+vk_1)T_{13}-y\rho_3+z\rho_2+T_{11}] \\
&\quad -\mathbf{i}_{\hat{1}}\cdot\mathbf{i}_{\hat{x}}+\mathbf{i}_{\hat{1}}\cdot\mathbf{i}_{\hat{x}}-1-\frac{z\rho_1}{\tau\hat{\tau}}[(u'-vk_3+wk_2)T_{21}+(v'+uk_3-wk_1)T_{22}+(w'-uk_2+vk_1)T_{23}-z\rho_1+T_{21}] \\
&\quad +\frac{y\rho_1}{\tau\hat{\tau}}[(u'-vk_3+wk_2)T_{31}+(v'+uk_3-wk_1)T_{32}+(w'-uk_2+vk_1)T_{33}+y\rho_1+T_{31}] \\
&= \frac{1+e-y\rho_3+z\rho_2}{\tau\hat{\tau}}[1+e-y\rho_3+z\rho_2]-1+\frac{z^2\rho_1^2+y^2\rho_1^2}{\tau\hat{\tau}} \\
&= \frac{\hat{\tau}}{\tau}-1.
\end{aligned} \tag{A8}$$

Expanding eqn (A8) and neglecting terms proportional to y^m and z^n, $m, n \geqslant 2$, we obtain

$$\varepsilon_{11} = e-y[\rho_3-(1+e)k_3]+z[\rho_2-(1+e)k_2]. \tag{A9a}$$

We note that k_1 does not appear in eqn (A9a) and hence the expression of the axial strain along the ξ direction is the same as that along the $\hat{\xi}$ direction [i.e. eqn (A9a)] because $\mathbf{i}_{\hat{1}} = \mathbf{i}_1$ if the influence of k_1 is neglected [see eqn (A7b)]. The factor $(1+e)$ in eqn (A9a) is due to the fact that ρ_2 and ρ_3 are not real curvatures [see eqn (19)] whereas k_2 and k_3 are real curvatures and ε_{11} is a strain defined with respect to the undeformed length. Assuming $1+e \simeq 1$, we rewrite eqn (A9a) as

$$\varepsilon_{11} = e-y(\rho_3-k_3)+z(\rho_2-k_2). \tag{A9b}$$

As pointed out in Section 3, the rigid-body translations and rotations do not produce any strains and the strains are due to relative displacements. Therefore, one can choose the observed cross section on the η–ζ plane to be fixed. Consequently, the displacements of an arbitrary point, which is very close to this cross-section, can be expressed as

$$\begin{aligned}
u_1(s,y,z,t) &= u_1^0(s,t)+z[\theta_2(s,t)-\theta_{20}(s)]-y[\theta_3(s,t)-\theta_{30}(s)] \\
u_2(s,y,z,t) &= u_2^0(s,t)-z[\theta_1(s,t)-\theta_{10}(s)] \\
u_3(s,y,z,t) &= u_3^0(s,t)+y[\theta_1(s,t)-\theta_{10}(s)],
\end{aligned} \tag{A10}$$

where u_1, u_2 and u_3 are local, strainable displacements with respect to the ξ, η and ζ axes, respectively; $u_i^0(s,t) \equiv u_i(s,0,0,t)$, $i = 1,2,3$; θ_1, θ_2 and θ_3 are the rotation angles of the observed cross-section with respect to the ξ, η and ζ axes, respectively; θ_{10}, θ_{20} and θ_{30} are the initial rotation angles of the observed cross-section with respect to the ξ, η and ζ axes, respectively.

Because the $\xi\eta\zeta$ is a local coordinate system attached to the observed cross-section and the unit vector $\mathbf{i}_1$ is tangent to the deformed reference axis, eqns (21) and (24) are valid. Because all the variables in eqn (A10) are zero for any point on the cross-section and only the first derivatives of eqn (A10) with respect to s, y and z are needed to calculate the strains, nonlinear terms are not needed in eqn (A10). Moreover, because the local displacement vector $\mathbf{U}$, which is given by

$$\mathbf{U} = u_1\mathbf{i}_1+u_2\mathbf{i}_2+u_3\mathbf{i}_3 \tag{A11}$$

is an infinitesimal vector defined with respect to the deformed coordinate system, it follows from eqns (A11), (21), (24) and (18a, b) that

$$\varepsilon_{11} = \frac{\partial\mathbf{U}}{\partial s}\cdot\mathbf{i}_1 = e+z(\rho_2-k_2)-y(\rho_3-k_3) \tag{A12}$$

which is the same as eqn (A9b). To obtain eqn (A9a), one needs to substitute $(1+e)\theta_{i0}$, $i = 1,2,3$ for θ_{i0} in eqn (A10).

Since the in-plane and out-of-plane warpings are relative displacements with respect to the flat surface on the η–ζ plane, they are essentially local displacements. Hence, warping displacements can be superposed on the local displacements shown in eqn (A10) [see eqn (20)], and the above method of deriving local strains is still valid even with nontrivial warpings.

APPENDIX B: THE EXPLICIT FORM OF STRUCTURAL TERMS IN EQNS (77b, c)

It follows from eqns (7b), (17a), (11b) and (12) that the transformation matrix $[T]$ in eqn (77b) is a function of u, v, w and ϕ given by

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ -T_{12} & T_{11}+T_{13}^2/(1+T_{11}) & -T_{12}T_{13}/(1+T_{11}) \\ -T_{13} & -T_{12}T_{13}/(1+T_{11}) & T_{11}+T_{12}^2/(1+T_{11}) \end{bmatrix}, \tag{B1a}$$

where

$$T_{11} = \frac{1+u'-vk_3+wk_2}{1+e}, \quad T_{12} = \frac{v'+uk_3-wk_1}{1+e}, \quad T_{13} = \frac{w'-uk_2+vk_1}{1+e} \tag{B1b}$$

$$e = \sqrt{(1+u'-vk_3+wk_2)^2+(v'+uk_3-wk_1)^2+(w'-uk_2+vk_1)^2}-1 \tag{B1c}$$

and the initial curvatures k_i are known functions of s, as shown in eqns (6c) and (5b). The force strain vector $\{\tilde{\psi}\}$ and the curvature matrix [K] in eqns (77b, c) are functions of u, v, w and ϕ and are obtained from eqns (26b), (18b) and (19) as

$$\{\tilde{\psi}\} = \{e, \gamma_6, \gamma_5, \rho_1-k_1, \rho_2-k_2, \rho_3-k_3, \gamma_6', \gamma_5'\}^T \tag{B2a}$$

$$[\mathrm{K}] = \begin{bmatrix} 0 & \rho_3 & -\rho_2 \\ -\rho_3 & 0 & \rho_1 \\ \rho_2 & -\rho_1 & 0 \end{bmatrix}, \tag{B2b}$$

where

$$\rho_1 = \sum_{i=1}^{3}(T_{2i}'T_{3i}+T_{1i}k_i), \quad \rho_2 = \sum_{i=1}^{3}(-T_{1i}'T_{3i}+T_{2i}k_i), \quad \rho_3 = \sum_{i=1}^{3}(T_{1i}'T_{2i}+T_{3i}k_i). \tag{B2c}$$

We note that eqns (B1a–c) and (B2b, c) are valid for all beam theories because they only describe the deformation of the reference axis ξ and are functions of u, v, w and ϕ.

The structural stiffness matrices $[A]$, $[B]$, $[D]$, $[E^1]$, $[E^2]$, $[E]$ and $[F]$ are obtained from eqns (51c) and (29) as

$$\begin{bmatrix} [A] & [B] \\ [B]^T & [D] \end{bmatrix} = \int_A [S]^T[\tilde{Q}][S]\,\mathrm{d}A$$

$$[E] = \begin{bmatrix} [E^1] \\ [E^2] \end{bmatrix} = \int_A [S]^T[y\{\tilde{Q}_1\}\ z\{\tilde{Q}_1\}]\,\mathrm{d}A, \quad [F] = \int_A \tilde{Q}_{11}\begin{bmatrix} y^2 & yz \\ yz & z^2 \end{bmatrix}\mathrm{d}A, \tag{B3}$$

where the 6×6 matrix $[S]$ is a function of s, y and z and it can be obtained from eqns (28b), (26b) and (27b). We note that the $[S]$ is a function of y and z only if the initial curvatures k_i are constants and the deformed curvatures ρ_i in $[\tilde{X}]$ and $[\tilde{\mathrm{K}}]$ [see eqn (26b)] are replaced by k_i.

APPENDIX C: THE NONLINEAR EULER–BERNOULLI THEORY FOR CURVED BEAMS IN HOVER

Substituting eqns (45) and (59) into eqn (30), using eqns (82c) and (81) and setting each of the coefficients of δu, δv, δw, $\delta\theta_1$, $\delta\theta_2$ and $\delta\theta_3$ equal to zero, we obtain the following equations of motion

$$[T]^T\left(\begin{Bmatrix} F_1' \\ F_2' \\ F_3' \end{Bmatrix} - [\mathrm{K}]\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}\right) = \begin{Bmatrix} A_u+\mu_1\dot{u} \\ A_v+\mu_2\dot{v} \\ A_w+\mu_3\dot{w} \end{Bmatrix} \tag{C1a}$$

$$M_1'-M_2\rho_3+M_3\rho_2 = A_{\theta_1}+\mu_4\dot{\phi} \tag{C1b}$$

$$F_3 = \frac{1}{1+e}(M_2'-M_3\rho_1+M_1\rho_3-A_{\theta_2}) \tag{C2a}$$

$$F_2 = \frac{-1}{1+e}(M_3'+M_2\rho_1-M_1\rho_2-A_{\theta_3}). \tag{C2b}$$

We point out here that the governing equations are the four equations in eqns (C1a, b) and eqns (C2a, b) represent the shear forces F_3 and F_2 in terms of stress resultants and moments. The boundary conditions for the beam are of the form

$$\begin{aligned}
\delta u = 0 &\quad \text{or} \quad F_1 T_{11} + F_2 T_{21} + F_3 T_{31} \quad \text{specified} \\
\delta v = 0 &\quad \text{or} \quad F_1 T_{12} + F_2 T_{22} + F_3 T_{32} \quad \text{specified} \\
\delta w = 0 &\quad \text{or} \quad F_1 T_{13} + F_2 T_{23} + F_3 T_{33} \quad \text{specified} \\
\delta \theta_1 = 0 &\quad \text{or} \quad M_1 \quad \text{specified} \\
\delta \theta_2 = 0 &\quad \text{or} \quad M_2 \quad \text{specified} \\
\delta \theta_3 = 0 &\quad \text{or} \quad M_3 \quad \text{specified.}
\end{aligned} \tag{C3}$$

It follows from eqns (27c), (29) and (80a) that

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} A_{11} B_{11} B_{12} B_{13} \\ B_{11} D_{11} D_{12} D_{13} \\ B_{12} D_{12} D_{22} D_{23} \\ B_{13} D_{13} D_{23} D_{33} \end{bmatrix} \begin{Bmatrix} e \\ \rho_1 - k_1 \\ \rho_2 - k_2 \\ \rho_3 - k_3 \end{Bmatrix}, \tag{C4a}$$

where

$$\begin{bmatrix} A_{11} B_{11} B_{12} B_{13} \\ B_{11} D_{11} D_{12} D_{13} \\ B_{12} D_{12} D_{22} D_{23} \\ B_{13} D_{13} D_{23} D_{33} \end{bmatrix} \equiv \int_A \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{15} y - \tilde{Q}_{16} z & \tilde{Q}_{11} z & -\tilde{Q}_{11} y \\ & \tilde{Q}_{55} y^2 + \tilde{Q}_{66} z^2 - 2\tilde{Q}_{65} yz & \tilde{Q}_{15} yz - \tilde{Q}_{16} z^2 & \tilde{Q}_{16} yz - \tilde{Q}_{15} y^2 \\ & & \tilde{Q}_{11} z^2 & -\tilde{Q}_{11} yz \\ \text{sym.} & & & \tilde{Q}_{11} y^2 \end{bmatrix} \mathrm{d}A. \tag{C4b}$$

We note that the explicit expressions of $[T]$, [K], ρ_i, and e are the same as those shown in eqns (B1a), (B2b), (B2c) and (B1c). Moreover, the expressions of F_3 and F_2 can be obtained from eqns (C2a, b).

For helicopter rotor blades in a hover condition, the matrix $[T^a]$ is a function of time, as shown in eqn (84b). Moreover, the matrix $[T^x]$ is a function of s only, as shown in eqns (5b), and the matrix $[T]$ is a function of s and t, as shown in eqn (B1a). It follows from eqns (46), (1f), (8a–c), (83), (84a) and (84c) that the inertial terms are

$$\begin{Bmatrix} A_u \\ A_v \\ A_w \end{Bmatrix} = I_{11}([T^x]\{Q_1\} + \{Q_2\}), \quad \begin{Bmatrix} A_{\theta_1} \\ A_{\theta_2} \\ A_{\theta_3} \end{Bmatrix} = \begin{Bmatrix} \bar{I}_{22}\dot{\omega}_1 - (I_{33} - I_{22})\omega_2\omega_3 \\ I_{33}\dot{\omega}_2 - (I_{22} - \bar{I}_{22})\omega_1\omega_3 \\ I_{22}\dot{\omega}_3 - (\bar{I}_{22} - I_{33})\omega_1\omega_2 \end{Bmatrix}, \tag{C5a}$$

where $\bar{I}_{22} \equiv I_{22} + I_{33}$ and

$$\{Q_1\} = \{-B\dot{\Omega} - A\Omega^2, A\dot{\Omega} - B\Omega^2, 0\}^T$$

$$\{Q_2\} = \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{Bmatrix} + \begin{bmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{bmatrix} \begin{Bmatrix} \dot{\omega}_1^x \\ \dot{\omega}_2^x \\ \dot{\omega}_3^x \end{Bmatrix} + 2 \begin{bmatrix} 0 & \dot{w} & -\dot{v} \\ -\dot{w} & 0 & \dot{u} \\ \dot{v} & -\dot{u} & 0 \end{bmatrix} \begin{Bmatrix} \omega_1^x \\ \omega_2^x \\ \omega_3^x \end{Bmatrix}$$
$$- \begin{bmatrix} 0 & \omega_3^x & -\omega_2^x \\ -\omega_3^x & 0 & \omega_1^x \\ \omega_2^x & -\omega_1^x & 0 \end{bmatrix} \begin{bmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{bmatrix} \begin{Bmatrix} \omega_1^x \\ \omega_2^x \\ \omega_3^x \end{Bmatrix}. \tag{C5b}$$